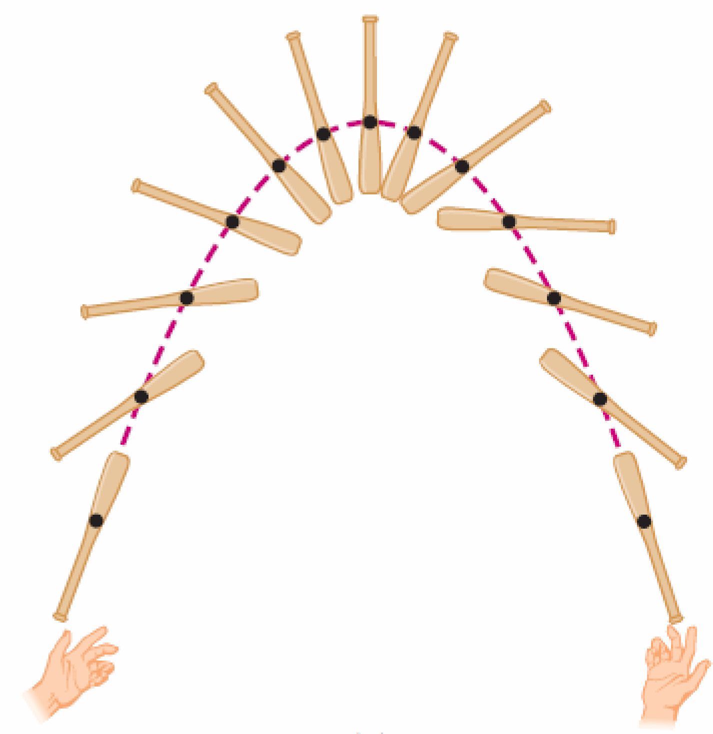
Chapter 9

Center of Mass

and

Linear Momentum

The center of mass of a system of particles is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.



The center of mass (black dot) of a baseball bat flipped into the air follows a parabolic path, but all other points of the bat follow more complicated curved paths.

9.2 The Center of Mass: A System of Particles

Consider a situation in which n particles are strung out along the x axis. Let the mass of the particles are m_1, m_2, ….m_n, and let them be located at x_1, x_2, …x_n respectively. Then if the total mass is $M = m_1 + m_2 + \ldots + m_n$, then the location of the center of mass, x_{com}, is

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{M}$$

$$= \frac{1}{M} \sum_{i=1}^{n} m_i x_i .$$

9.2 The Center of Mass: A System of Particles

In 3-D, the locations of the center of mass are given by:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \qquad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \qquad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i.$$

The position of the center of mass can be expressed as:

$$\vec{r}_{com} = x_{com}\hat{i} + y_{com}\hat{j} + z_{com}\hat{k}.$$

$$\Downarrow$$

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i,$$

9.2 The Center of Mass: Solid Body

In the case of a solid body, the "particles" become differential mass elements dm, the sums become integrals, and the coordinates of the center of mass are defined as

$$x_{com} = \frac{1}{M}\int x\,dm, \qquad y_{com} = \frac{1}{M}\int y\,dm, \qquad z_{com} = \frac{1}{M}\int z\,dm$$

where M is the mass of the object.

If the object has uniform density, r, defined as: $\rho = \frac{dm}{dV} = \frac{M}{V}$,

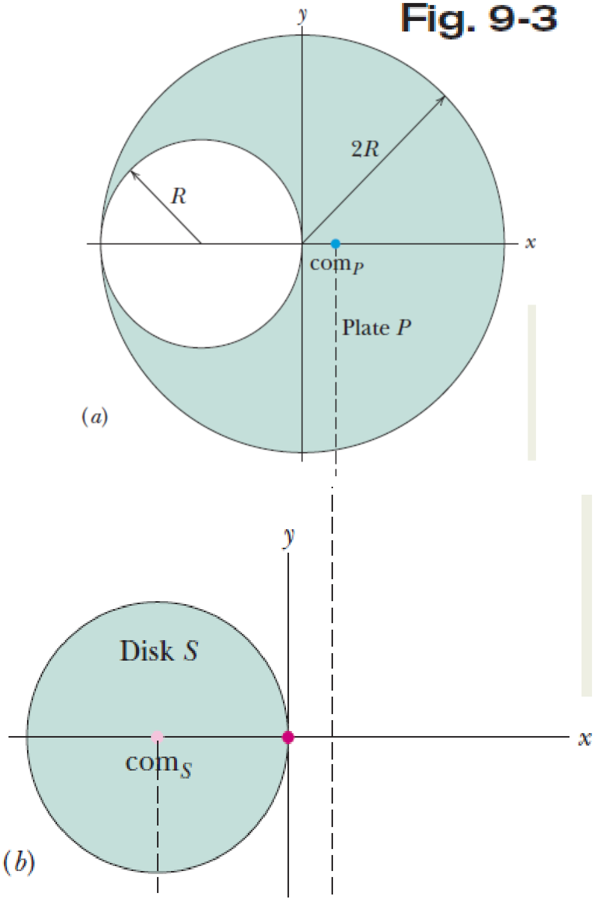
Then $$x_{com} = \frac{1}{V}\int x\,dV, \qquad y_{com} = \frac{1}{V}\int y\,dV, \qquad z_{com} = \frac{1}{V}\int z\,dV.$$

Where V is the volume of the object.

Sample problem, COM

Figure 9-3*a* shows a uniform metal plate *P* of radius 2*R* from which a disk of radius *R* has been stamped out (removed) in an assembly line. The disk is shown in Fig. 9-3b. Using the *xy* coordinate system shown, locate the center of mass com_P of the remaining plate.

Fig. 9-3



Calculations: First, put the stamped-out disk (call it disk S) back into place to form the original composite plate (call it plate C). Because of its circular symmetry, the center of mass com_S for disk S is at the center of S, at x =-R. Similarly, the center of mass com_C for composite plate C is at the center of C, at the origin. Assume that mass mS of disk S is concentrated in a particle at x_S =-R, and mass m_P is concentrated in a particle at x_P. Next treat these two particles as a two particle system, and find their center of mass x_{S+P}.

$$x_{S+P} = \frac{m_S x_S + m_P x_P}{m_S + m_P}.$$

Next note that the combination of disk S and plate P is composite plate C. Thus, the position x_{S+P} of com_{S+P} must coincide with the position x_C of com_C, which is at the origin; so $x_{S+P} = x_C = 0$.

$$x_P = -x_S \frac{m_S}{m_P}.$$

But,

$$\frac{m_S}{m_P} = \frac{\text{density}_S}{\text{density}_P} \times \frac{\text{thickness}_S}{\text{thickness}_P} \times \frac{\text{area}_S}{\text{area}_P}.$$

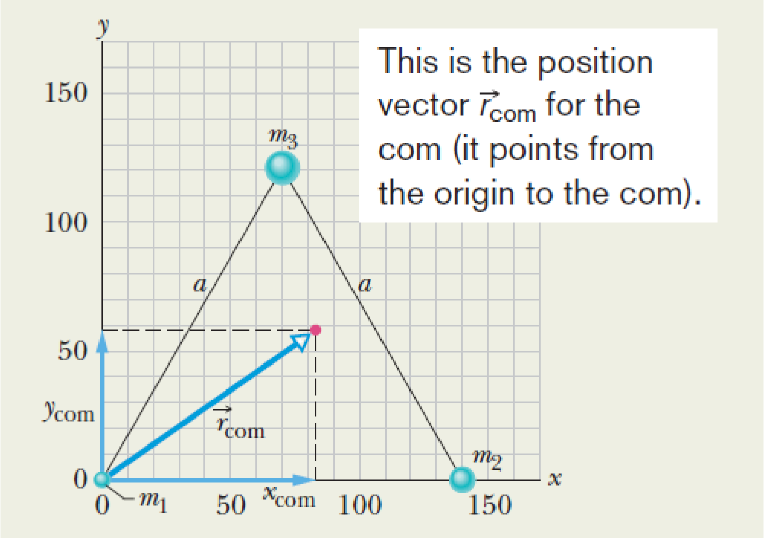
$$\frac{m_S}{m_P} = \frac{\text{area}_S}{\text{area}_P} = \frac{\text{area}_S}{\text{area}_C - \text{area}_S} = \frac{\pi R^2}{\pi(2R)^2 - \pi R^2} = \frac{1}{3}.$$

and x_S=-R

$$x_P = \tfrac{1}{3}R.$$

Sample problem, COM of 3 particles

Three particles of masses $m_1 = 1.2$ kg, $m_2 = 2.5$ kg, and $m_3 = 3.4$ kg form an equilateral triangle of edge length $a = 140$ cm. Where is the center of mass of this system?



We are given the following data:

Particle	Mass (kg)	x (cm)	y (cm)
1	1.2	0	0
2	2.5	140	0
3	3.4	70	120

The total mass M of the system is 7.1 kg.

The coordinates of the center of mass are therefore:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(140\text{ cm}) + (3.4\text{ kg})(70\text{ cm})}{7.1\text{ kg}}$$

$$= 83\text{ cm} \qquad \text{(Answer)}$$

and $$y_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{M}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(0) + (3.4\text{ kg})(120\text{ cm})}{7.1\text{ kg}}$$

$$= 58\text{ cm}. \qquad \text{(Answer)}$$

Note that the $z_{com} = 0$.

9.3: Newton's 2nd Law for a System of Particles

The vector equation that governs the motion of the center of mass of such a system of particles is:

$$\vec{F}_{net} = M\vec{a}_{com} \qquad \text{(system of particles).}$$

$$F_{net,x} = Ma_{com,x} \qquad F_{net,y} = Ma_{com,y} \qquad F_{net,z} = Ma_{com,z}.$$

Note that:

1. F_{net} is the net force of all external forces that act on the system. Forces on one part of the system from another part of the system (internal forces) are not included

2. M is the total mass of the system. M remains constant, and the system is said to be closed.

3. a_{com} is the acceleration of the center of mass of the system.

The internal forces of the explosion cannot change the path of the com.

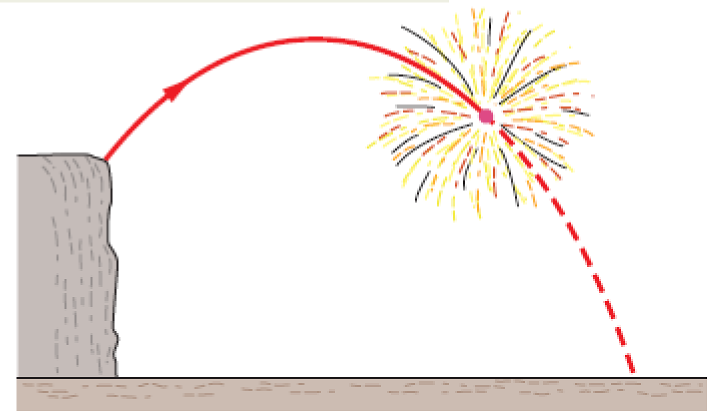


Fig. 9-5 A fireworks rocket explodes in flight. In the absence of air drag, the center of mass of the fragments would continue to follow the original parabolic path, until fragments began to hit the ground.

9.3: Newton's 2nd Law for a System of Particles: Proof of final result

➢For a system of *n* particles, $M\vec{r}_{com} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n,$

where M is the total mass, and $\mathbf{r}_i$ are the position vectors of the masses m_i.

➢Differentiating, $M\vec{v}_{com} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n.$

where the **v** vectors are velocity vectors.

➢This leads to $M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n.$

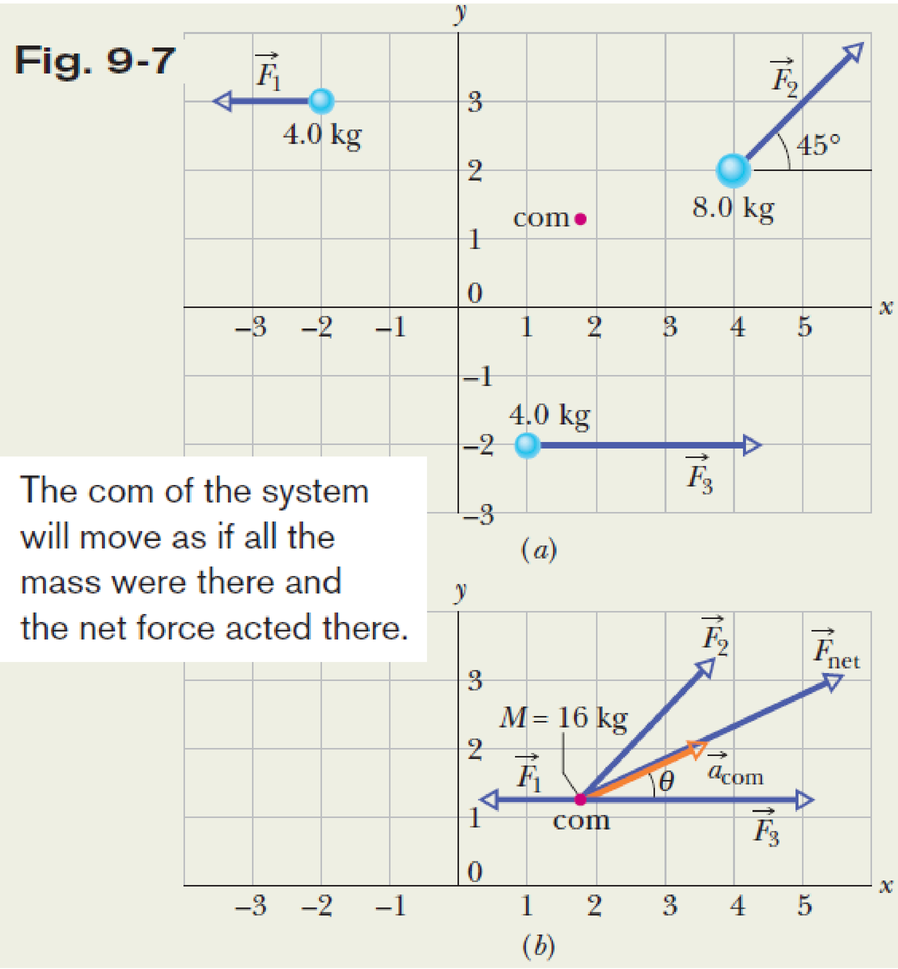
➢Finally, $M\vec{a}_{com} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n.$

What remains on the right hand side is the vector sum of all the external forces that act on the system, while the internal forces cancel out by Newton's 3rd Law.

Sample problem: motion of the com of 3 particles

The three particles in Fig. 9-7*a* are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 6.0$ N, $F_2 = 12$ N, and $F_3 = 14$ N. What is the acceleration of the center of mass of the system, and in what direction does it move?

Fig. 9-7



Calculations: Applying Newton's second law to the center of mass,

$$\vec{F}_{net} = M\vec{a}_{com}$$

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{com}$$

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}.$$

$$a_{com,x} = \frac{F_{1x} + F_{2x} + F_{3x}}{M}$$

$$= \frac{-6.0\text{ N} + (12\text{ N})\cos 45° + 14\text{ N}}{16\text{ kg}} = 1.03\text{ m/s}^2.$$

Along the y axis, we have

$$a_{com,y} = \frac{F_{1y} + F_{2y} + F_{3y}}{M}$$

$$= \frac{0 + (12\text{ N})\sin 45° + 0}{16\text{ kg}} = 0.530\text{ m/s}^2.$$

From these components, we find that $\vec{a}_{com}$ has the magnitude

$$a_{com} = \sqrt{(a_{com,x})^2 + (a_{com,y})^2}$$

$$= 1.16\text{ m/s}^2 \approx 1.2\text{ m/s}^2 \qquad \text{(Answer)}$$

and the angle (from the positive direction of the x axis)

$$\theta = \tan^{-1}\frac{a_{com,y}}{a_{com,x}} = 27°. \qquad \text{(Answer)}$$

9.4: Linear Momentum

DEFINITION:

$$\vec{p} = m\vec{v} \qquad \text{(linear momentum of a particle)}$$

in which m is the mass of the particle and **v** is its velocity.

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}.$$

Manipulating this equation:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

(Newton's 2^{nd} Law)

9.5: The Linear Momentum of a System of Particles

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

$$\vec{P} = M\vec{v}_{com} \quad \text{(linear momentum, system of particles),}$$

$$\Downarrow$$

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \quad \text{(system of particles),}$$

9.6: Collision and Impulse

The collision of a ball with a bat collapses part of the ball. *(Photo by Harold E. Edgerton. ©The Harold and Esther Edgerton Family Trust, courtesy of Palm Press, Inc.)*

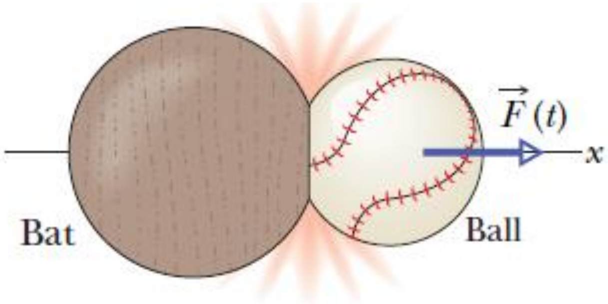


Fig. 9-8 Force $\vec{F}(t)$ acts on a ball as the ball and a bat collide.

In this case, the collision is brief, and the ball experiences a force that is great enough to slow, stop, or even reverse its motion.

The figure depicts the collision at one instant. The ball experiences a force F(t) that varies during the collision and changes the linear momentum of the ball.

9.6: Collision and Impulse

The change in linear momentum is related to the force by Newton's second law written in the form

$$\vec{F} = d\vec{p}/dt.$$

$$\longrightarrow \quad \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\, dt.$$

$$\longrightarrow \quad \vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\, dt \qquad \text{(impulse defined).}$$

The right side of the equation is a measure of both the magnitude and the duration of the collision force, and is called the ***impulse of the collision*, J.**

9.6: Collision and Impulse

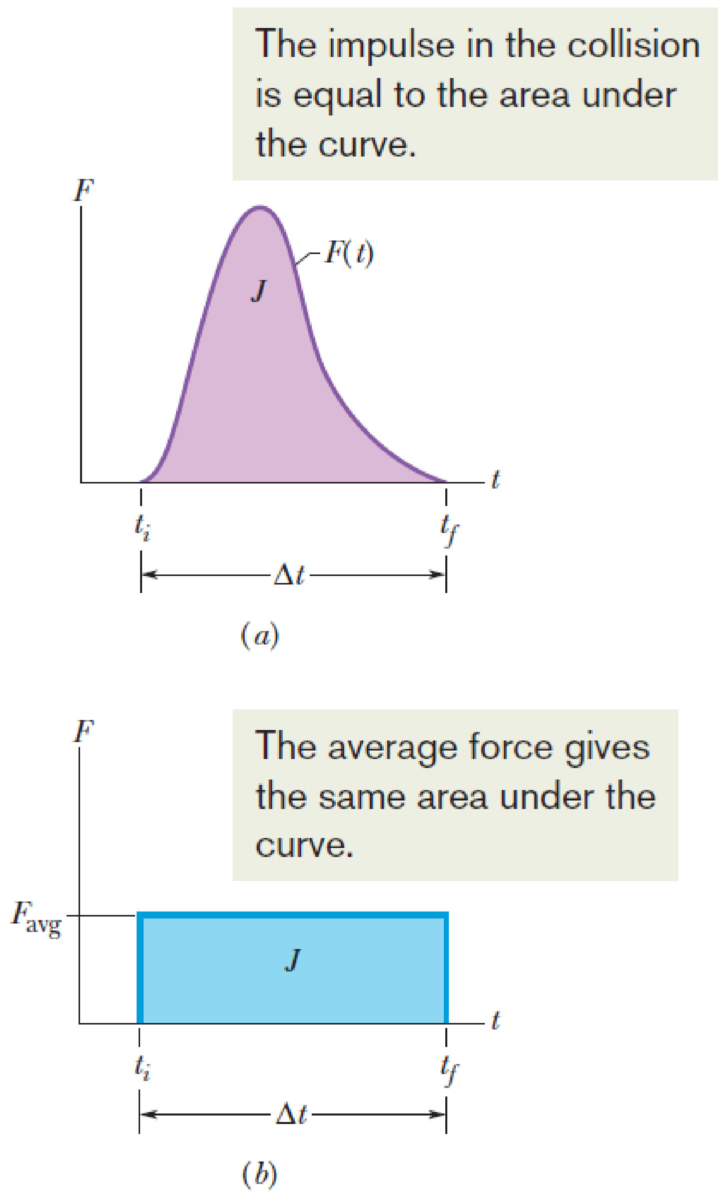


Fig. 9-9 (*a*) The curve shows the magnitude of the time-varying force $F(t)$ that acts on the ball in the collision of Fig. 9-8. The area under the curve is equal to the magnitude of the impulse $\vec{J}$ on the ball in the collision. (*b*) The height of the rectangle represents the average force F_{avg} acting on the ball over the time interval Δt. The area within the rectangle is equal to the area under the curve in (*a*) and thus is also equal to the magnitude of the impulse $\vec{J}$ in the collision.

Instead of the ball, one can focus on the bat. At any instant, Newton's third law says that the force on the bat has the same magnitude but the opposite direction as the force on the ball.

That means that the impulse on the bat has the same magnitude but the opposite direction as the impulse on the ball.

9.6: Collision and Impulse: Series of Collisions

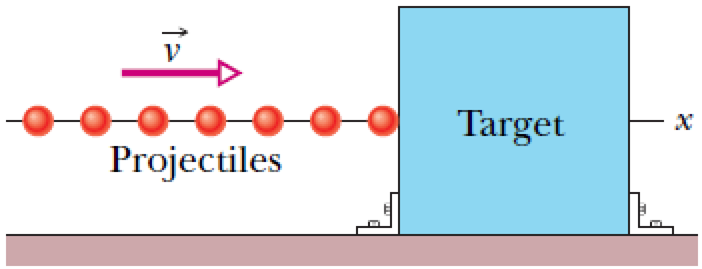


Fig. 9-10 A steady stream of projectiles, with identical linear momenta, collides with a target, which is fixed in place. The average force F_{avg} on the target is to the right and has a magnitude that depends on the rate at which the projectiles collide with the target or, equivalently, the rate at which mass collides with the target.

Let n be the number of projectiles that collide in a time interval Δt.

Each projectile has initial momentum mv and undergoes a change Δp in linear momentum because of the collision.

The total change in linear momentum for n projectiles during interval Δt is nΔp. The resulting impulse on the target during Δt is along the x axis and has the same magnitude of nΔp but is in the opposite direction.

$$J = -n\,\Delta p,$$

$$F_{avg} = \frac{J}{\Delta t} = -\frac{n}{\Delta t}\Delta p = -\frac{n}{\Delta t} m\,\Delta v.$$

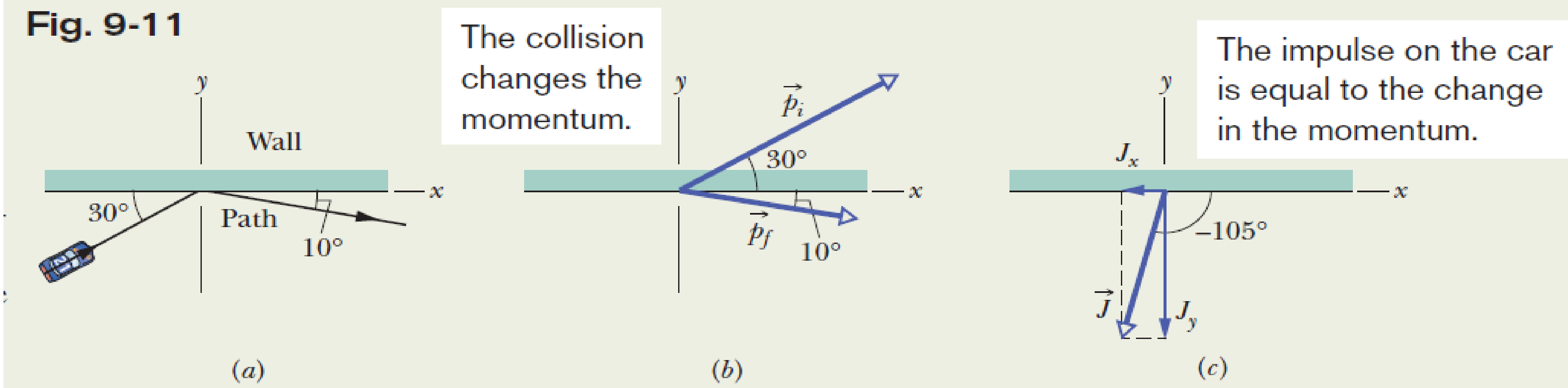
In time interval Δt, an amount of mass Δm =nm collides with the target.

$$F_{avg} = -\frac{\Delta m}{\Delta t}\Delta v.$$

Sample problem: 2-D impulse

Race car–wall collision. Figure 9-11*a* is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70$ m/s along a straight line at 30° from the wall. Just after the collision, he is traveling at speed $v_f = 50$ m/s along a straight line at 10° from the wall. His mass m is 80 kg.

Fig. 9-11



(a) What is the impulse $\vec{J}$ on the driver due to the collision?

Calculations: Figure 9-11*b* shows the driver's momentum $\vec{p}_i$ before the collision (at angle 30° from the positive x direction) and his momentum $\vec{p}_f$ after the collision (at angle −10°).

$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i).$$

x component: Along the x axis we have

$$\begin{aligned} J_x &= m(v_{fx} - v_{ix}) \\ &= (80\text{ kg})[(50\text{ m/s})\cos(-10°) - (70\text{ m/s})\cos 30°] \\ &= -910\text{ kg}\cdot\text{m/s}. \end{aligned}$$

y component: Along the y axis,

$$\begin{aligned} J_y &= m(v_{fy} - v_{iy}) \\ &= (80\text{ kg})[(50\text{ m/s})\sin(-10°) - (70\text{ m/s})\sin 30°] \\ &= -3495\text{ kg}\cdot\text{m/s} \approx -3500\text{ kg}\cdot\text{m/s}. \end{aligned}$$

Impulse: The impulse is then

$$\vec{J} = (-910\hat{i} - 3500\hat{j})\text{ kg}\cdot\text{m/s}, \qquad \text{(Answer)}$$

which means the impulse magnitude is

$$J = \sqrt{J_x^2 + J_y^2} = 3616\text{ kg}\cdot\text{m/s} \approx 3600\text{ kg}\cdot\text{m/s}.$$

The angle of $\vec{J}$ is given by

$$\theta = \tan^{-1}\frac{J_y}{J_x}, \qquad \text{(Answer)}$$

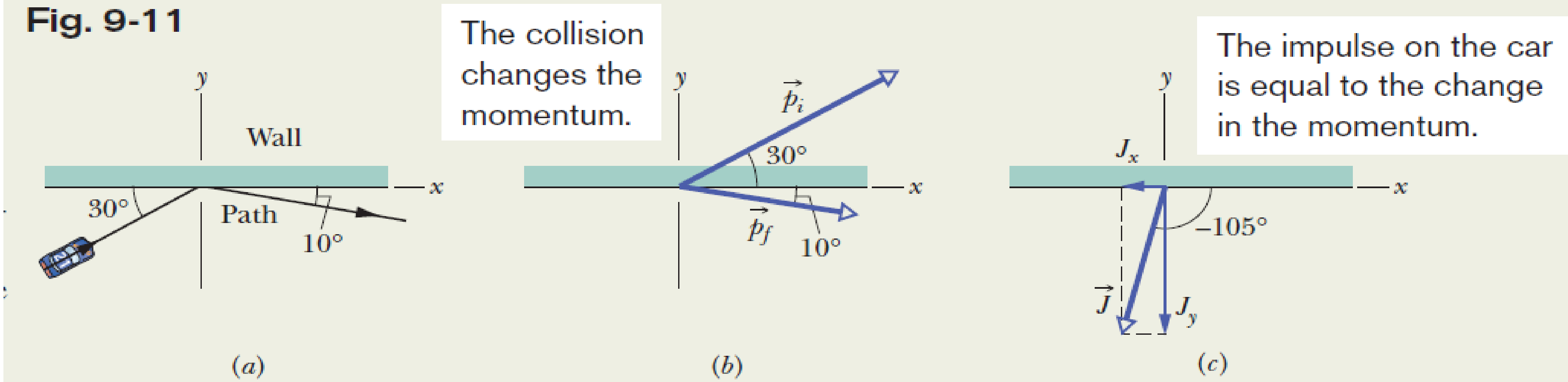
which a calculator evaluates as 75.4°. Recall that the physically correct result of an inverse tangent might be the displayed answer plus 180°. We can tell which is correct here by drawing the components of $\vec{J}$ (Fig. 9-11*c*). We find that θ is actually 75.4° + 180° = 255.4°, which we can write as

$$\theta = -105°. \qquad \text{(Answer)}$$

Sample problem: 2-D impulse, cont.

Race car–wall collision. Figure 9-11*a* is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70$ m/s along a straight line at 30° from the wall. Just after the collision, he is traveling at speed $v_f = 50$ m/s along a straight line at 10° from the wall. His mass m is 80 kg.

Fig. 9-11



(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

Calculations: We have

$$F_{avg} = \frac{J}{\Delta t} = \frac{3616 \text{ kg} \cdot \text{m/s}}{0.014 \text{ s}}$$

$$= 2.583 \times 10^5 \text{ N} \approx 2.6 \times 10^5 \text{ N}. \qquad \text{(Answer)}$$

Using $F = ma$ with $m = 80$ kg, you can show that the magnitude of the driver's average acceleration during the collision is about 3.22×10^3 m/s^2 = 329g, which is fatal.

Surviving: Mechanical engineers attempt to reduce the chances of a fatality by designing and building racetrack walls with more "give," so that a collision lasts longer. For example, if the collision here lasted 10 times longer and the other data remained the same, the magnitudes of the average force and average acceleration would be 10 times less and probably survivable.

9.7: Conservation of Linear Momentum

If no net external force acts on a system of particles, the total linear momentum, **P**, of the system cannot change.

$$\vec{P} = \text{constant} \qquad \text{(closed, isolated system).}$$

If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

Sample problem: 1-D explosion

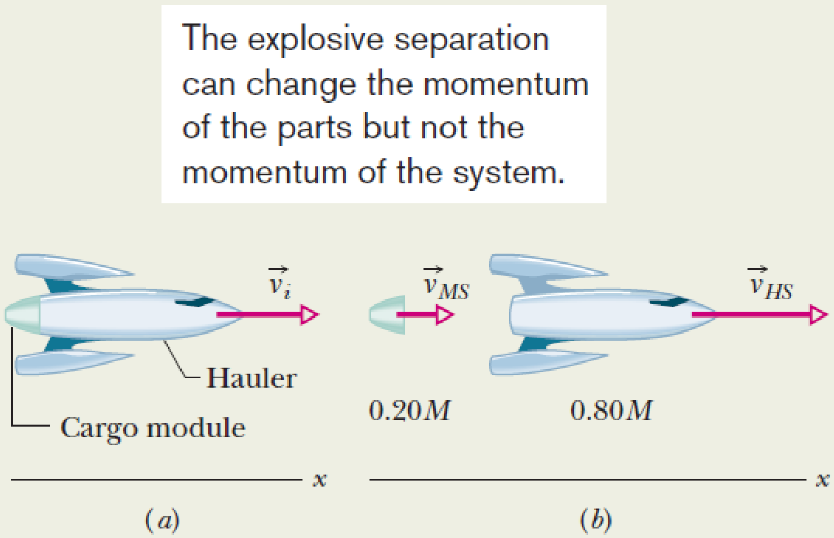
One-dimensional explosion: Figure 9-12*a* shows a space hauler and cargo module, of total mass M, traveling along an x axis in deep space. They have an initial velocity $\vec{v}_i$ of magnitude 2100 km/h relative to the Sun. With a small explosion, the hauler ejects the cargo module, of mass $0.20M$ (Fig. 9-12*b*). The hauler then travels 500 km/h faster than the module along the x axis; that is, the relative speed v_{rel} between the hauler and the module is 500 km/h. What then is the velocity $\vec{v}_{HS}$ of the hauler relative to the Sun?

KEY IDEA

Because the hauler–module system is closed and isolated, its total linear momentum is conserved; that is,

$$\vec{P}_i = \vec{P}_f, \tag{9-44}$$

Fig. 9-12



where the subscripts i and f refer to values before and after the ejection, respectively.

Calculations: Because the motion is along a single axis, we can write momenta and velocities in terms of their x components, using a sign to indicate direction. Before the ejection, we have

$$P_i = Mv_i. \tag{9-45}$$

Let v_{MS} be the velocity of the ejected module relative to the Sun. The total linear momentum of the system after the ejection is then

$$P_f = (0.20M)v_{MS} + (0.80M)v_{HS}, \tag{9-46}$$

where the first term on the right is the linear momentum of the module and the second term is that of the hauler.

We do not know the velocity v_{MS} of the module relative to the Sun, but we can relate it to the known velocities with

$$\begin{pmatrix}\text{velocity of}\\ \text{hauler relative}\\ \text{to Sun}\end{pmatrix} = \begin{pmatrix}\text{velocity of}\\ \text{hauler relative}\\ \text{to module}\end{pmatrix} + \begin{pmatrix}\text{velocity of}\\ \text{module relative}\\ \text{to Sun}\end{pmatrix}.$$

In symbols, this gives us

$$v_{HS} = v_{rel} + v_{MS} \tag{9-47}$$

or

$$v_{MS} = v_{HS} - v_{rel}.$$

Substituting this expression for v_{MS} into Eq. 9-46, and then substituting Eqs. 9-45 and 9-46 into Eq. 9-44, we find

$$Mv_i = 0.20M(v_{HS} - v_{rel}) + 0.80Mv_{HS},$$

which gives us

$$v_{HS} = v_i + 0.20v_{rel},$$

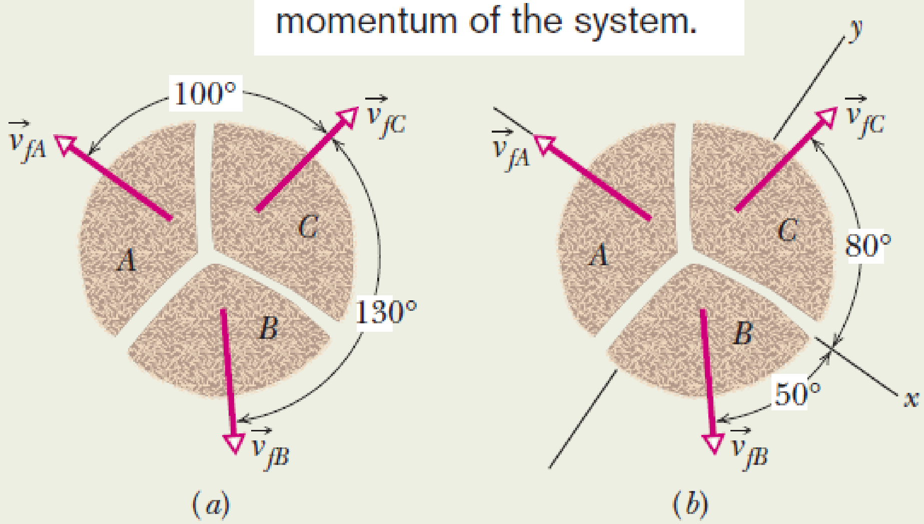
or

$$v_{HS} = 2100 \text{ km/h} + (0.20)(500 \text{ km/h})$$
$$= 2200 \text{ km/h}. \quad \text{(Answer)}$$

Sample problem: 2-D explosion

Two-dimensional explosion: A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor. An overhead view is shown in Fig. 9-13*a*. Piece C, with mass $0.30M$, has final speed $v_{fC} = 5.0$ m/s.

Fig. 9-13 The explosive separation can change the momentum of the parts but not the momentum of the system.



(a) What is the speed of piece B, with mass $0.20M$?

Calculations: To get started, we superimpose an xy coordinate system as shown in Fig. 9-13*b*, with the negative direction of the x axis coinciding with the direction of $\vec{v}_{fA}$. The x axis is at 80° with the direction of $\vec{v}_{fC}$ and 50° with the direction of $\vec{v}_{fB}$.

$$P_{iy} = P_{fy},$$

where subscript i refers to the initial value (before the explosion), and subscript y refers to the y component of $\vec{P}_i$ or $\vec{P}_f$.

The component P_{iy} of the initial linear momentum is zero, because the coconut is initially at rest. To get an expression for P_{fy}, we find the y component of the final linear momentum of each piece, using the y-component version of Eq. 9-22 ($p_y = mv_y$):

$$p_{fA,y} = 0,$$
$$p_{fB,y} = -0.20Mv_{fB,y} = -0.20Mv_{fB} \sin 50°,$$
$$p_{fC,y} = 0.30Mv_{fC,y} = 0.30Mv_{fC} \sin 80°.$$

(Note that $p_{fA,y} = 0$ because of our choice of axes.) Equation 9-48 can now be written as

$$P_{iy} = P_{fy} = p_{fA,y} + p_{fB,y} + p_{fC,y}.$$

Then, with $v_{fC} = 5.0$ m/s, we have

$$0 = 0 - 0.20Mv_{fB} \sin 50° + (0.30M)(5.0 \text{ m/s}) \sin 80°,$$

from which we find

$$v_{fB} = 9.64 \text{ m/s} \approx 9.6 \text{ m/s}. \qquad \text{(Answer)}$$

(b) What is the speed of piece A?

$$p_{fA,x} = -0.50Mv_{fA},$$
$$p_{fB,x} = 0.20Mv_{fB,x} = 0.20Mv_{fB} \cos 50°,$$
$$p_{fC,x} = 0.30Mv_{fC,x} = 0.30Mv_{fC} \cos 80°.$$
$$P_{ix} = P_{fx} = p_{fA,x} + p_{fB,x} + p_{fC,x}.$$

Then, with $v_{fC} = 5.0$ m/s and $v_{fB} = 9.64$ m/s, we have

$$0 = -0.50Mv_{fA} + 0.20M(9.64 \text{ m/s}) \cos 50° + 0.30M(5.0 \text{ m/s}) \cos 80°,$$

from which we find

$$v_{fA} = 3.0 \text{ m/s}. \qquad \text{(Answer)}$$

9.8: Momentum and Kinetic Energy in Collisions

In a closed and isolated system, if there are two colliding bodies, and the total kinetic energy is unchanged by the collision, then the kinetic energy of the system is conserved (it is the same before and after the collision). Such a collision is called an ***elastic collision*.**

If during the collision, some energy is always transferred from kinetic energy to other forms of energy, such as thermal energy or energy of sound, then the kinetic energy of the system is not conserved. Such a collision is called an ***inelastic collision*.**

9.9: Inelastic collisions in One Dimension

Here is the generic setup for an inelastic collision.

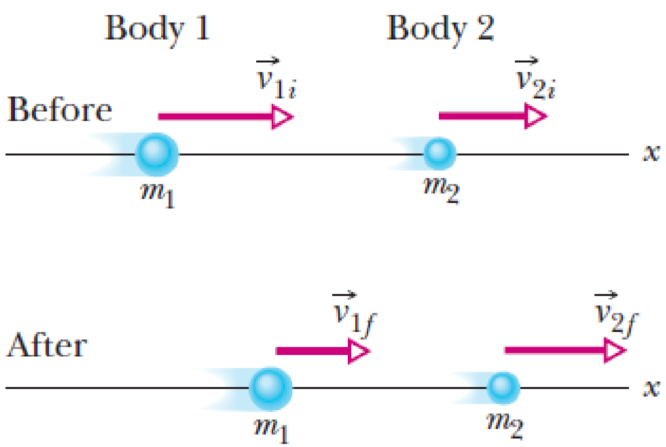


Fig. 9-14 Bodies 1 and 2 move along an x axis, before and after they have an inelastic collision.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}.$$

In a completely inelastic collision, the bodies stick together.

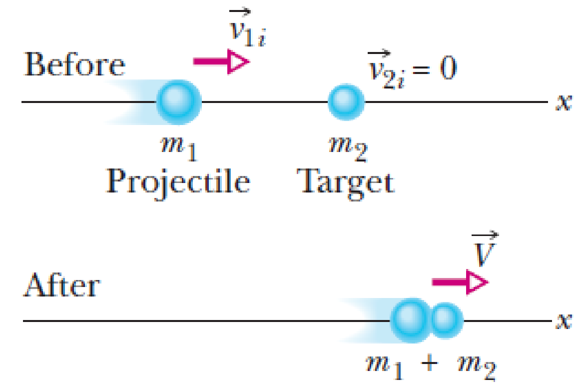


Fig. 9-15 A completely inelastic collision between two bodies. Before the collision, the body with mass m_2 is at rest and the body with mass m_1 moves directly toward it. After the collision, the stuck-together bodies move with the same velocity $\vec{V}$.

$$m_1 v_{1i} = (m_1 + m_2)V$$

$$V = \frac{m_1}{m_1 + m_2} v_{1i}.$$

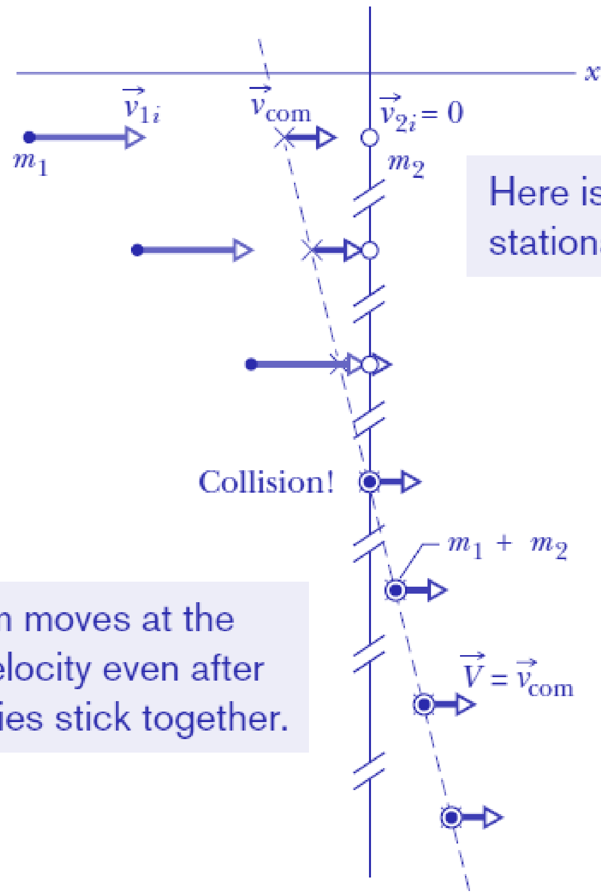
9.9: Inelastic collisions in 1-D: Velocity of Center of Mass

The com of the two bodies is between them and moves at a constant velocity.

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}.$$

Here is the incoming projectile.

Here is the stationary target.



The com moves at the same velocity even after the bodies stick together.

Fig. 9-16 Some freeze frames of a two-body system, which undergoes a completely inelastic collision. The system's center of mass is shown in each freeze-frame. The velocity v_{com} of the center of mass is unaffected by the collision.

Because the bodies stick together after the collision, their common velocity V must be equal to v_{com}.

Sample problem: conservation of momentum

The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. The version shown in Fig. 9-17 consists of a large block of wood of mass $M = 5.4$ kg, hanging from two long cords. A bullet of mass $m = 9.5$ g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance $h = 6.3$ cm before the pendulum comes momentarily to rest at the end of its arc. What is the speed of the bullet just prior to the collision?

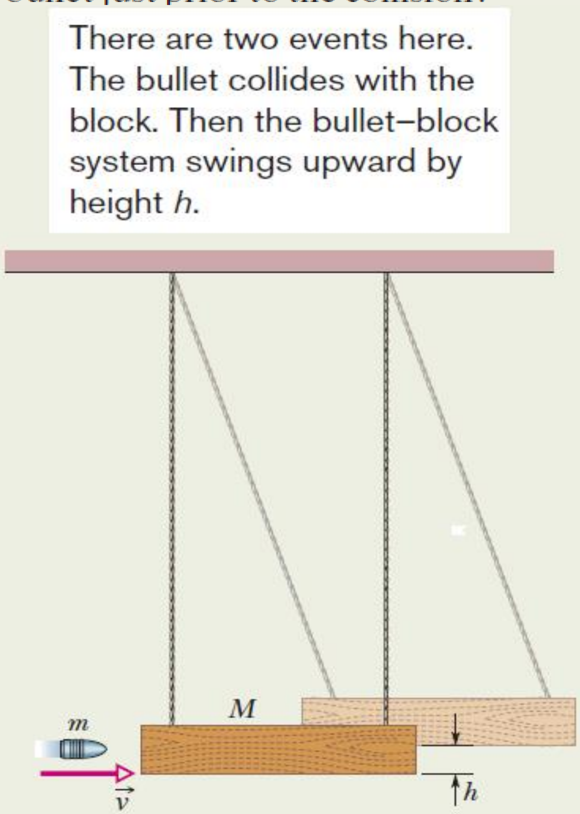


Fig. 9-17 A ballistic pendulum, used to measure the speeds of bullets.

The collision within the bullet– block system is so brief. Therefore:

(1) During the collision, the gravitational force on the block and the force on the block from the cords are still balanced. Thus, during the collision, the net external impulse on the bullet–block system is zero. Therefore, the system is isolated and its total linear momentum is conserved.

(2) The collision is one-dimensional in the sense that the direction of the bullet and block just after the collision is in the bullet's original direction of motion.

$$V = \frac{m}{m + M} v.$$

As the bullet and block now swing up together, the mechanical energy of the bullet–block–Earth system is conserved:

$$\tfrac{1}{2}(m + M)V^2 = (m + M)gh.$$

Combining steps:

$$v = \frac{m + M}{m}\sqrt{2gh}$$

$$= \left(\frac{0.0095 \text{ kg} + 5.4 \text{ kg}}{0.0095 \text{ kg}}\right)\sqrt{(2)(9.8 \text{ m/s}^2)(0.063 \text{ m})}$$

$$= 630 \text{ m/s.} \qquad \text{(Answer)}$$

9.10: Elastic collisions in One Dimension

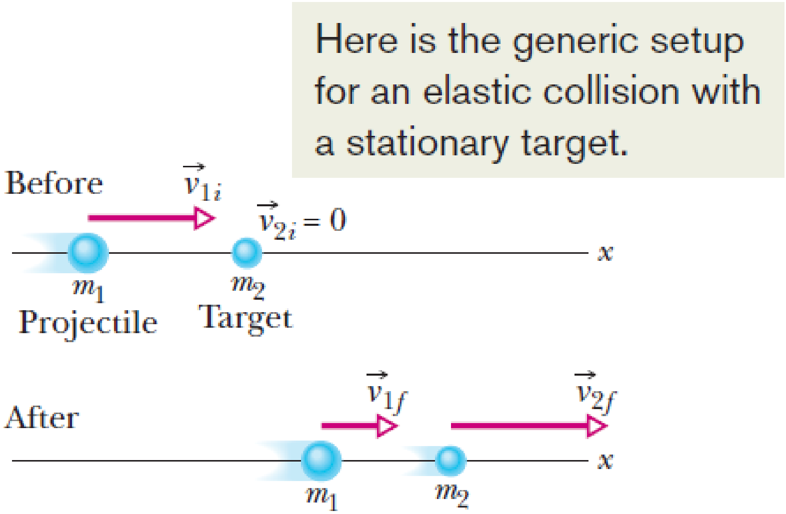


Fig. 9-18 Body 1 moves along an x axis before having an elastic collision with body 2, which is initially at rest. Both bodies move along that axis after the collision.

In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.

9.10: Elastic collisions in One Dimension: Stationary Target

Here is the generic setup for an elastic collision with a stationary target.

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \qquad \text{(linear momentum)}.$$

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \qquad \text{(kinetic energy)}.$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

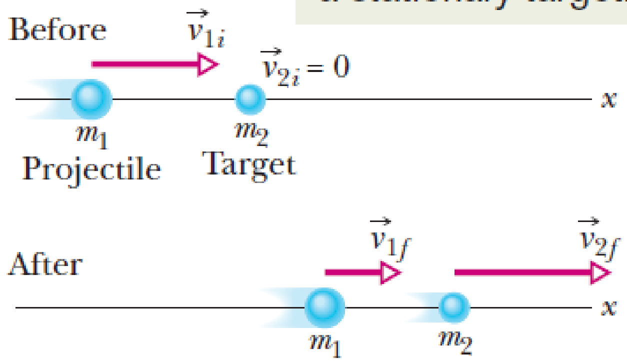


Fig. 9-18 Body 1 moves along an x axis before having an elastic collision with body 2, which is initially at rest. Both bodies move along that axis after the collision.

9.10: Elastic collisions in One Dimension: Moving Target

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f},$$

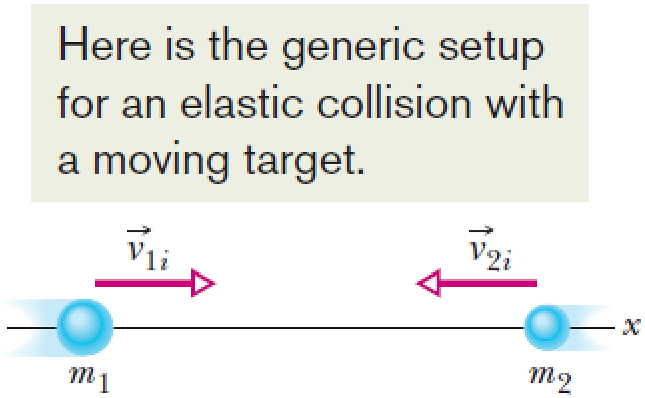


Fig. 9-19 Two bodies headed for a one-dimensional elastic collision.

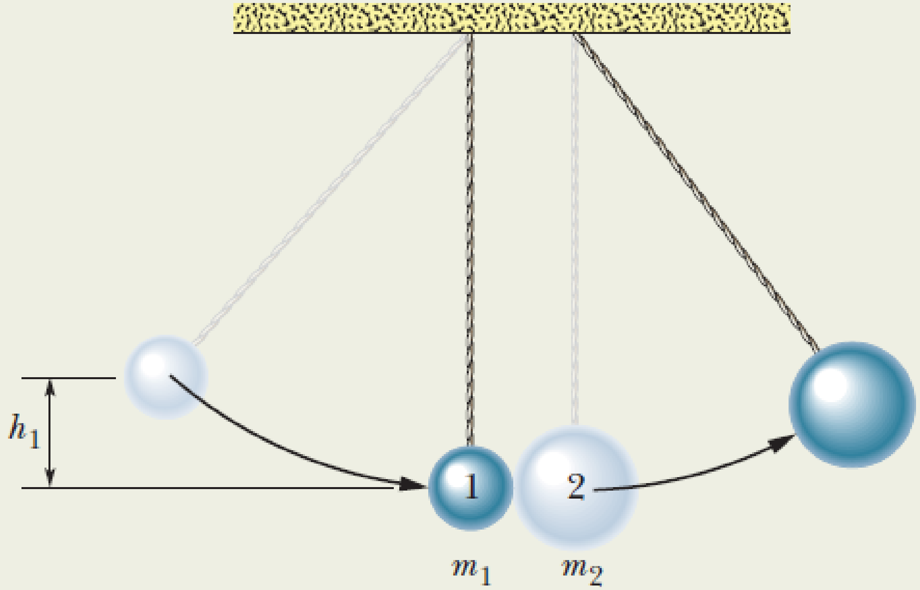
$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2.$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}.$$

Sample problem: Two pendulums

Two metal spheres, suspended by vertical cords, initially just touch, as shown in Fig. 9-20. Sphere 1, with mass $m_1 = 30$ g, is pulled to the left to height $h_1 = 8.0$ cm, and then released from rest. After swinging down, it undergoes an elastic collision with sphere 2, whose mass $m_2 = 75$ g. What is the velocity v_{1f} of sphere 1 just after the collision?



Step 1: As sphere 1 swings down, the mechanical energy of the sphere–Earth system is conserved. (The mechanical energy is not changed by the force of the cord on sphere 1 because that force is always directed perpendicular to the sphere's direction of travel.)

Calculation: Let's take the lowest level as our reference level of zero gravitational potential energy. Then the kinetic energy of sphere 1 at the lowest level must equal the gravitational potential energy of the system when sphere 1 is at height h_1. Thus,

$$\tfrac{1}{2}m_1v_{1i}^2 = m_1gh_1,$$

which we solve for the speed v_{1i} of sphere 1 just before the collision:

$$v_{1i} = \sqrt{2gh_1} = \sqrt{(2)(9.8\text{ m/s}^2)(0.080\text{ m})}$$
$$= 1.252\text{ m/s}.$$

Step 2: Here we can make two assumptions in addition to the assumption that the collision is elastic. First, we can assume that the collision is one-dimensional because the motions of the spheres are approximately horizontal from just before the collision to just after it. Second, because the collision is so

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}v_{1i}$$
$$= \frac{0.030\text{ kg} - 0.075\text{ kg}}{0.030\text{ kg} + 0.075\text{ kg}}(1.252\text{ m/s})$$
$$= -0.537\text{ m/s} \approx -0.54\text{ m/s}. \qquad \text{(Answer)}$$

The minus sign tells us that sphere 1 moves to the left just after the collision.

9.11: Collisions in Two Dimensions

A glancing collision that conserves both momentum and kinetic energy.

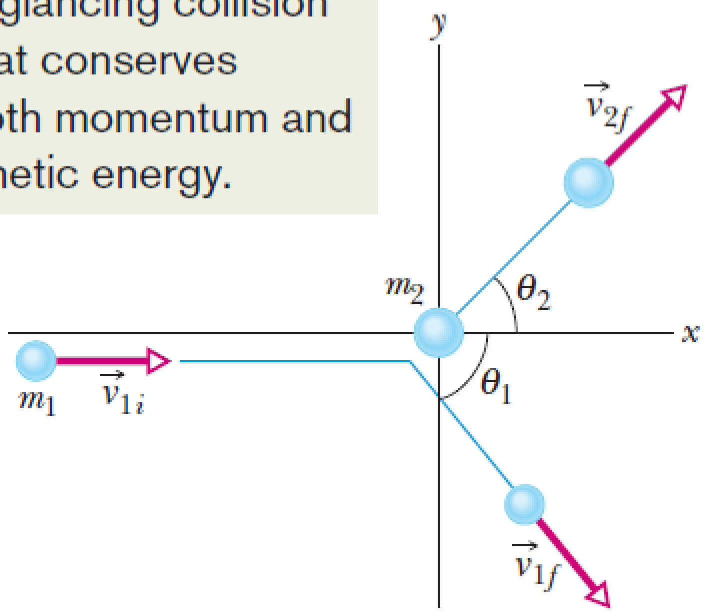


Fig. 9-21 An elastic collision between two bodies in which the collision is not head-on. The body with mass m_2 (the target) is initially at rest.

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}.$$

If elastic, $K_{1i} + K_{2i} = K_{1f} + K_{2f}.$

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2,$$

$$0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2.$$

Also, $\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

9.12: Systems with Varying Mass: A Rocket

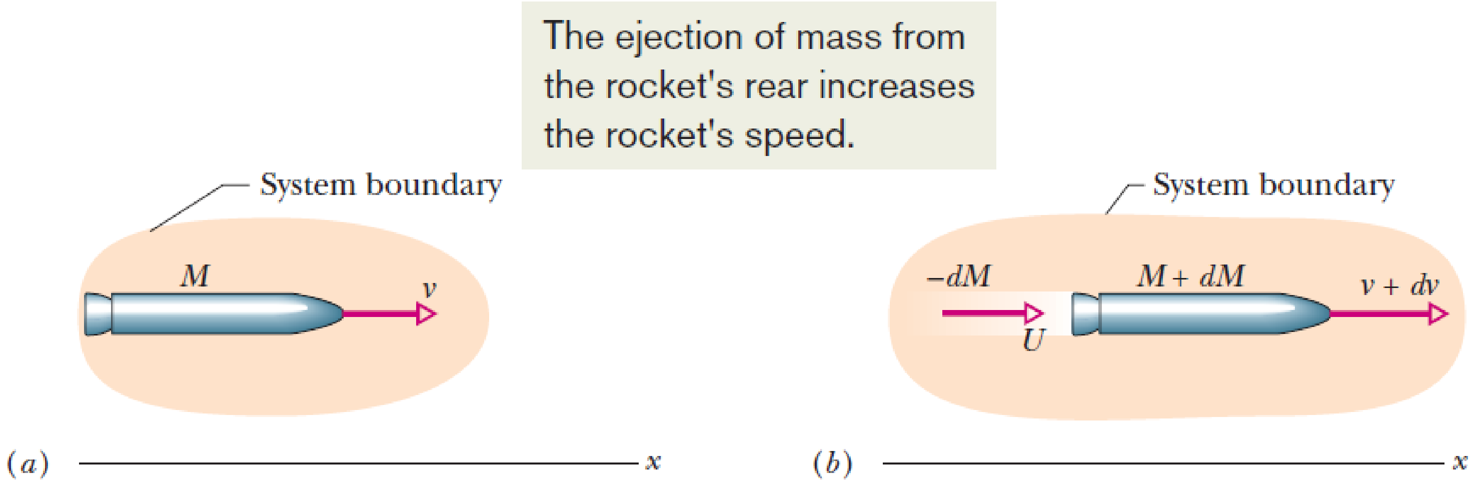


Fig. 9-22 (*a*) An accelerating rocket of mass *M* at time *t*, as seen from an inertial reference frame. (*b*) The same but at time *t* + *dt*. The exhaust products released during interval *dt* are shown.

The system here consists of the rocket and the exhaust products released during interval dt. The system is closed and isolated, so the linear momentum of the system must be conserved during dt, where the subscripts i and f indicate the values at the beginning and end of time interval dt.

$$P_i = P_f, \quad \Longrightarrow \quad Mv = -dM\,U + (M + dM)(v + dv)$$

$$\begin{pmatrix}\text{velocity of rocket}\\ \text{relative to frame}\end{pmatrix} = \begin{pmatrix}\text{velocity of rocket}\\ \text{relative to products}\end{pmatrix} + \begin{pmatrix}\text{velocity of products}\\ \text{relative to frame}\end{pmatrix}$$

$$(v + dv) = v_{\text{rel}} + U,$$

$$U = v + dv - v_{\text{rel}}.$$

$$\Longrightarrow \quad -\frac{dM}{dt} v_{\text{rel}} = M\frac{dv}{dt}. \quad \Longrightarrow \quad Rv_{\text{rel}} = Ma$$

9.12: Systems with Varying Mass: Finding the velocity

$$dv = -v_{\text{rel}} \frac{dM}{M}.$$

$$\int_{v_i}^{v_f} dv = -v_{\text{rel}} \int_{M_i}^{M_f} \frac{dM}{M},$$

in which M_i is the initial mass of the rocket and M_f its final mass. Evaluating the integrals then gives

$$v_f - v_i = v_{\text{rel}} \ln \frac{M_i}{M_f}$$

for the increase in the speed of the rocket during the change in mass from M_i to M_f .

Sample Problem: rocket engine, thrust, acceleration

A rocket whose initial mass M_i is 850 kg consumes fuel at the rate $R = 2.3$ kg/s. The speed v_{rel} of the exhaust gases relative to the rocket engine is 2800 m/s. What thrust does the rocket engine provide?

KEY IDEA

Thrust T is equal to the product of the fuel consumption rate R and the relative speed v_{rel} at which exhaust gases are expelled, as given by Eq. 9-87.

Calculation: Here we find

$$T = Rv_{rel} = (2.3 \text{ kg/s})(2800 \text{ m/s})$$
$$= 6440 \text{ N} \approx 6400 \text{ N}. \qquad \text{(Answer)}$$

(b) What is the initial acceleration of the rocket?

KEY IDEA

We can relate the thrust T of a rocket to the magnitude a of the resulting acceleration with $T = Ma$, where M is the rocket's mass. However, M decreases and a increases as fuel is consumed. Because we want the initial value of a here, we must use the intial value M_i of the mass.

Calculation: We find

$$a = \frac{T}{M_i} = \frac{6440 \text{ N}}{850 \text{ kg}} = 7.6 \text{ m/s}^2. \qquad \text{(Answer)}$$

To be launched from Earth's surface, a rocket must have an initial acceleration greater than $g = 9.8$ m/s². That is, it must be greater than the gravitational acceleration at the surface. Put another way, the thrust T of the rocket engine must exceed the initial gravitational force on the rocket, which here has the magnitude $M_i g$, which gives us

$$(850 \text{ kg})(9.8 \text{ m/s}^2) = 8330 \text{ N}.$$

Because the acceleration or thrust requirement is not met (here $T = 6400$ N), our rocket could not be launched from Earth's surface by itself; it would require another, more powerful, rocket.