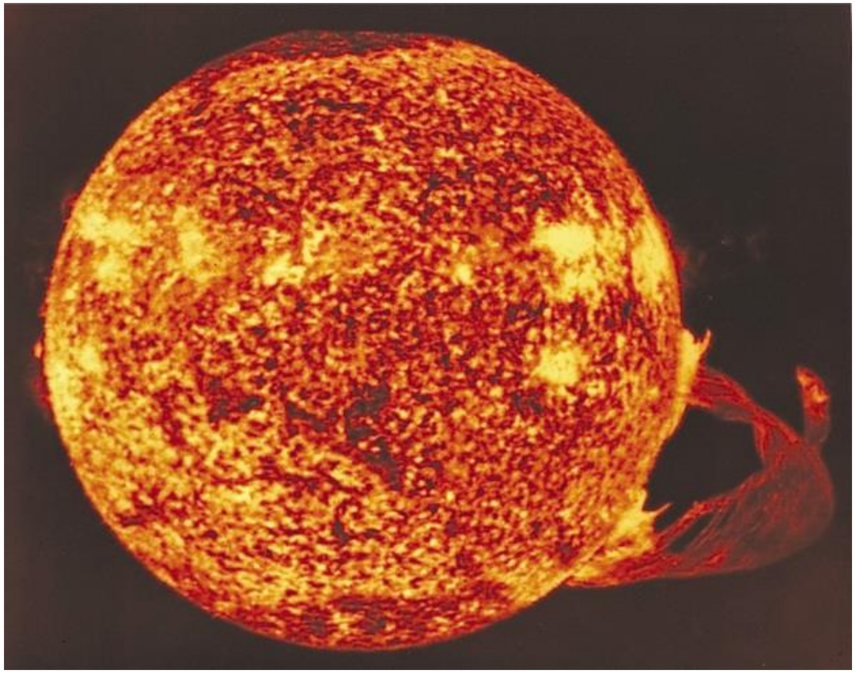
Chapter 31

Electromagnetic Oscillations and Alternating Current



31.2: LC Oscillations, Qualitatively:

In RC and RL circuits the charge, current, and potential difference grow and decay exponentially.

On the contrary, in an LC circuit, the charge, current, and potential difference vary sinusoidally with period T and angular frequency ω.

The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations.**

31.2: LC Oscillations, Qualitatively:

The energy stored in the electric field of the capacitor at any time is $U_E = \frac{q^2}{2C}$, where q is the charge on the capacitor at that time.

The energy stored in the magnetic field of the inductor at any time is $U_B = \frac{Li^2}{2}$, where i is the current through the inductor at that time.

As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.

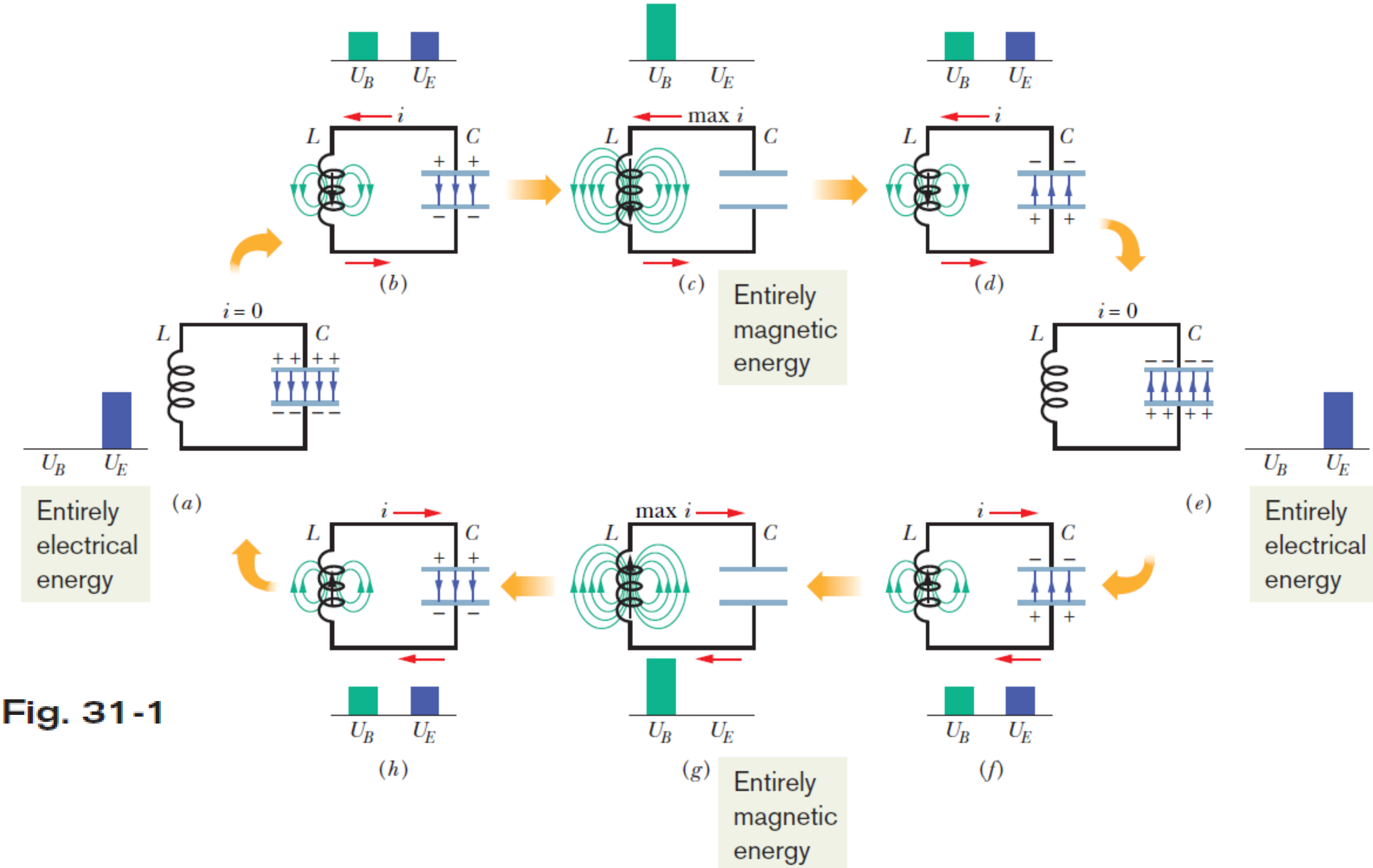


Fig. 31-1

31.2: LC Oscillations:

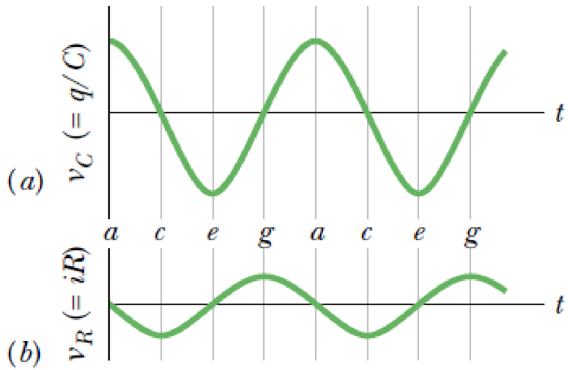
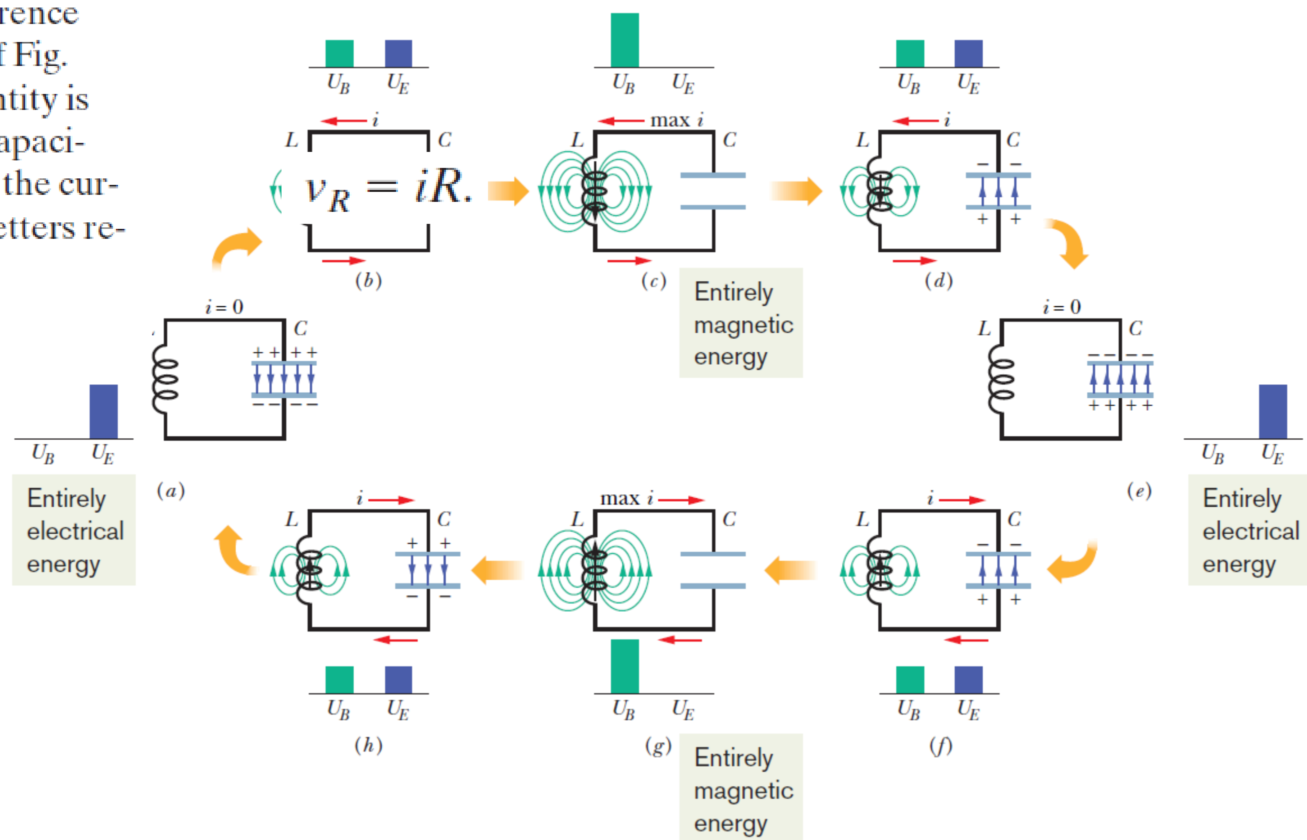


Fig. 31-2 (*a*) The potential difference across the capacitor of the circuit of Fig. 31-1 as a function of time. This quantity is proportional to the charge on the capacitor. (*b*) A potential proportional to the current in the circuit of Fig. 31-1. The letters refer to the correspondingly labeled oscillation stages in Fig. 31-1.

The time-varying potential difference (or *voltage*) v_C that exists across the capacitor C is

$$v_C = \left(\frac{1}{C}\right) q,$$

To measure the current, we can connect a small resistance R in series with the capacitor and inductor and measure the time-varying potential difference v_R across it: $v_R = iR.$



31.3: The Electrical-Mechanical Analogy:

One can make an analogy between the oscillating *LC* system and an oscillating block–spring system.

Two kinds of energy are involved in the block–spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block.

Here we have the following analogies:

$$q \text{ corresponds to } x, \qquad 1/C \text{ corresponds to } k,$$
$$i \text{ corresponds to } v, \quad \text{and} \quad L \text{ corresponds to } m.$$

Table 31-1

Comparison of the Energy in Two Oscillating Systems

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
	$v = dx/dt$		$i = dq/dt$

The angular frequency of oscillation for an ideal (resistanceless) *LC* is:

$$\omega = \frac{1}{\sqrt{LC}} \qquad (LC \text{ circuit}).$$

31.4: LC Oscillations, Quantitatively:

The Block-Spring Oscillator:

$$U = U_b + U_s = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2,$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0.$$

$$m\frac{d^2x}{dt^2} + kx = 0 \longrightarrow x = X\cos(\omega t + \phi) \quad \text{(displacement)}$$

The LC Oscillator:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C},$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0.$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad \text{(}LC\text{ oscillations).}$$

$$q = Q\cos(\omega t + \phi) \quad \text{(charge),} \longrightarrow i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi) \quad \text{(current).}$$

$$I = \omega Q, \longrightarrow \boxed{i = -I\sin(\omega t + \phi).}$$

31.4: LC Oscillations, Quantitatively:

Angular Frequencies:

$$\frac{d^2q}{dt^2} = -\omega^2 Q \cos(\omega t + \phi).$$

But $$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad (LC \text{ oscillations}).$$

$$\Rightarrow \quad -L\omega^2 Q \cos(\omega t + \phi) + \frac{1}{C} Q \cos(\omega t + \phi) = 0.$$

$$\Rightarrow \quad \boxed{\omega = \frac{1}{\sqrt{LC}}.}$$

31.4: LC Oscillations, Quantitatively:

The electrical energy stored in the *LC* circuit at time *t* is,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi).$$

The magnetic energy is:

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2 Q^2 \sin^2(\omega t + \phi).$$

But

$$\omega = \frac{1}{\sqrt{LC}} \qquad (LC \text{ circuit}).$$

Therefore

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi).$$

Note that

- The maximum values of U_E and U_B are both $Q^2/2C$.
- At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
- When U_E is maximum, U_B is zero, and conversely.

The electrical and magnetic energies vary but the total is constant.

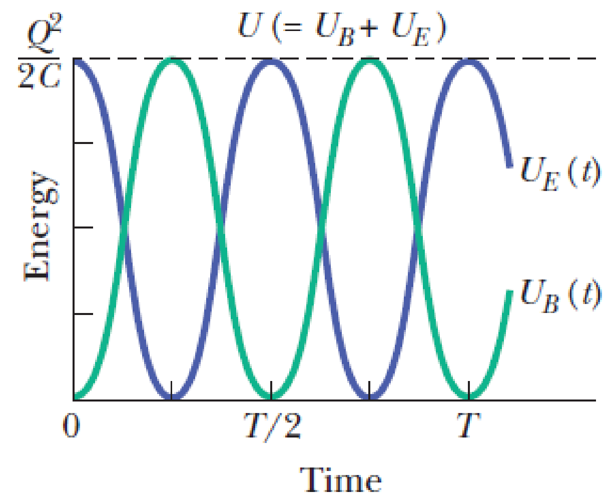


Fig. 31-4 The stored magnetic energy and electrical energy in the circuit of Fig. 31-1 as a function of time. Note that their sum remains constant. *T* is the period of oscillation.

Example, LC oscillator, potential charge, rate of current change

A 1.5 μF capacitor is charged to 57 V by a battery, which is then removed. At time $t = 0$, a 12 mH coil is connected in series with the capacitor to form an LC oscillator (Fig. 31-1).

(a) What is the potential difference $v_L(t)$ across the inductor as a function of time?

Calculations: At any time t during the oscillations, the loop rule and Fig. 31-1 give us

$$v_L(t) = v_C(t); \qquad (31\text{-}18)$$

that is, the potential difference v_L across the inductor must always be equal to the potential difference v_C across the capacitor, so that the net potential difference around the circuit is zero. Thus, we will find $v_L(t)$ if we can find $v_C(t)$, and we can find $v_C(t)$ from $q(t)$ with Eq. 25-1 ($q = CV$).

Because the potential difference $v_C(t)$ is maximum when the oscillations begin at time $t = 0$, the charge q on the capacitor must also be maximum then. Thus, phase constant ϕ must be zero; so Eq. 31-12 gives us

$$q = Q \cos \omega t. \qquad (31\text{-}19)$$

$$\frac{q}{C} = \frac{Q}{C} \cos \omega t,$$

$$v_C = V_C \cos \omega t. \qquad (31\text{-}20)$$

$$v_L = V_C \cos \omega t. \qquad (31\text{-}21)$$

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012\ \text{H})(1.5 \times 10^{-6}\ \text{F})]^{0.5}}$$

$$= 7454\ \text{rad/s} \approx 7500\ \text{rad/s}.$$

Thus, Eq. 31-21 becomes

$$v_L = (57\ \text{V}) \cos(7500\ \text{rad/s})t. \qquad \text{(Answer)}$$

(b) What is the maximum rate $(di/dt)_{max}$ at which the current i changes in the circuit?

Calculations: Taking the derivative, we have

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q \sin \omega t) = -\omega^2 Q \cos \omega t.$$

We can simplify this equation by substituting CV_C for Q (because we know C and V_C but not Q) and $1/\sqrt{LC}$ for ω according to Eq. 31-4. We get

$$\frac{di}{dt} = -\frac{1}{LC} CV_C \cos \omega t = -\frac{V_C}{L} \cos \omega t.$$

This tells us that the current changes at a varying (sinusoidal) rate, with its maximum rate of change being

$$\frac{V_C}{L} = \frac{57\ \text{V}}{0.012\ \text{H}} = 4750\ \text{A/s} \approx 4800\ \text{A/s}. \qquad \text{(Answer)}$$

31.5: Damped Oscillations in an RLC Circuit:

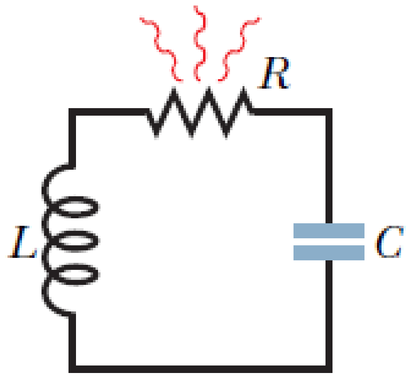


Fig. 31-5 A series *RLC* circuit. As the charge contained in the circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, damping (decreasing the amplitude of) the oscillations.

31.5: Damped Oscillations in an RLC Circuit:

Analysis:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}.$$

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \qquad (RLC \text{ circuit}),$$

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi),$$

Where $\omega' = \sqrt{\omega^2 - (R/2L)^2},$

And $\omega = 1/\sqrt{LC}$

$$U_E = \frac{q^2}{2C} = \frac{[Qe^{-Rt/2L}\cos(\omega' t + \phi)]^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \phi)$$

Example, Damped RLC Circuit:

Damped *RLC* circuit: charge amplitude

A series *RLC* circuit has inductance $L = 12$ mH, capacitance $C = 1.6\ \mu$F, and resistance $R = 1.5\ \Omega$ and begins to oscillate at time $t = 0$.

(a) At what time t will the amplitude of the charge oscillations in the circuit be 50% of its initial value? (Note that we do not know that initial value.)

KEY IDEA

The amplitude of the charge oscillations decreases exponentially with time t: According to Eq. 31-25, the charge amplitude at any time t is $Qe^{-Rt/2L}$, in which Q is the amplitude at time $t = 0$.

Calculations: We want the time when the charge amplitude has decreased to $0.50Q$, that is, when

$$Qe^{-Rt/2L} = 0.50Q.$$

We can now cancel Q (which also means that we can answer the question without knowing the initial charge). Taking the natural logarithms of both sides (to eliminate the exponential function), we have

$$-\frac{Rt}{2L} = \ln 0.50.$$

Solving for t and then substituting given data yield

$$t = -\frac{2L}{R}\ln 0.50 = -\frac{(2)(12 \times 10^{-3}\ \text{H})(\ln 0.50)}{1.5\ \Omega}$$

$$= 0.0111\ \text{s} \approx 11\ \text{ms}. \qquad \text{(Answer)}$$

(b) How many oscillations are completed within this time?

KEY IDEA

The time for one complete oscillation is the period $T = 2\pi/\omega$, where the angular frequency for LC oscillations is given by Eq. 31-4 ($\omega = 1/\sqrt{LC}$).

Calculation: In the time interval $\Delta t = 0.0111$ s, the number of complete oscillations is

$$\frac{\Delta t}{T} = \frac{\Delta t}{2\pi\sqrt{LC}}$$

$$= \frac{0.0111\ \text{s}}{2\pi[(12 \times 10^{-3}\ \text{H})(1.6 \times 10^{-6}\ \text{F})]^{1/2}} \approx 13.$$

(Answer)

Thus, the amplitude decays by 50% in about 13 complete oscillations. This damping is less severe than that shown in Fig. 31-3, where the amplitude decays by a little more than 50% in one oscillation.

31.6: Alternating Current:

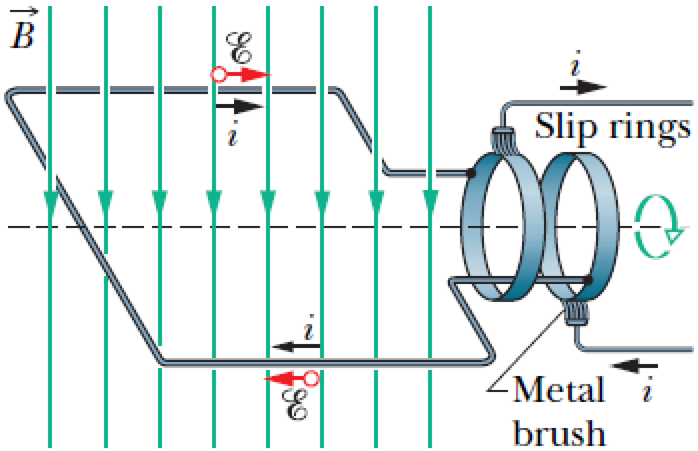


Fig. 31-6 The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which the ring slips as the loop (and it) rotates.

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t.$$

$$i = I \sin(\omega_d t - \phi),$$

ω_d is called the driving angular frequency, and I is the amplitude of the driven current.

31.7: Forced Oscillations:

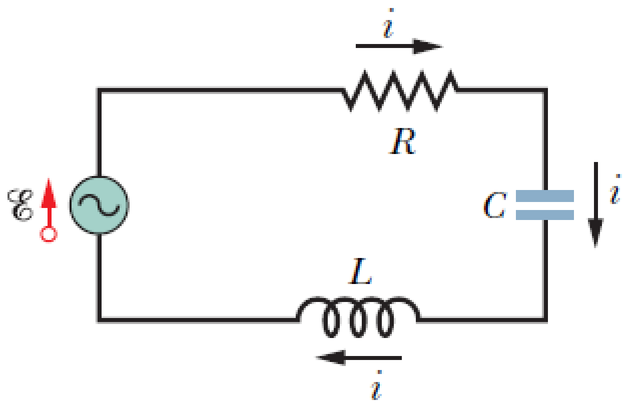


Fig. 31-7 A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

Whatever the natural angular frequency ω of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

31.8: Three Simple Circuits
i. A Resistive Load:

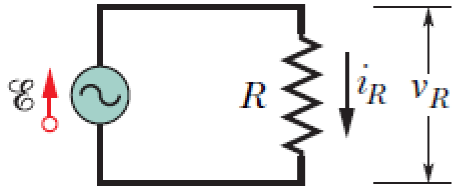


Fig. 31-8 A resistor is connected across an alternating-current generator.

$$\mathscr{E} - v_R = 0.$$

$$v_R = \mathscr{E}_m \sin \omega_d t. = V_R \sin \omega_d t.$$

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t.$$

$$= I_R \sin(\omega_d t - \phi),$$

For a purely resistive load, the phase constant $\phi = 0°$.

31.8: Three Simple Circuits:
i. A Resistive Load:

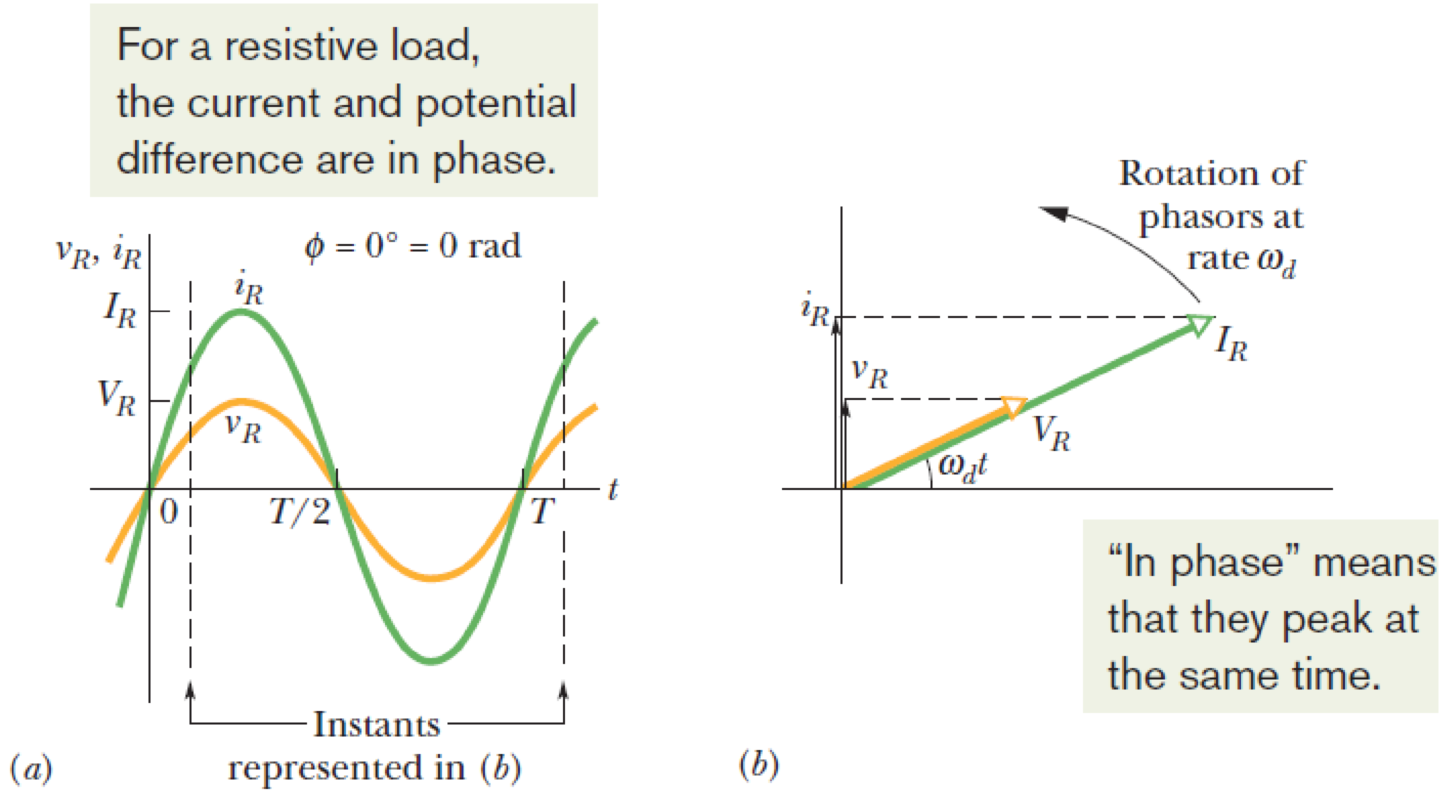


Fig. 31-9 (*a*) The current i_R and the potential difference v_R across the resistor are plotted on the same graph, both versus time *t*. They are in phase and complete one cycle in one period *T*. (*b*) A phasor diagram shows the same thing as (*a*).

Example, Purely resistive load: potential difference and current

In Fig. 31-8, resistance R is 200 Ω and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What is the potential difference $v_R(t)$ across the resistance as a function of time t, and what is the amplitude V_R of $v_R(t)$?

KEY IDEA

In a circuit with a purely resistive load, the potential difference $v_R(t)$ across the resistance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here we have $v_R(t) = \mathscr{E}(t)$ and $V_R = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we can write

$$V_R = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_R(t)$, we use Eq. 31-28 to write

$$v_R(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t \qquad (31\text{-}34)$$

and then substitute $\mathscr{E}_m = 36.0$ V and

$$\omega_d = 2\pi f_d = 2\pi(60 \text{ Hz}) = 120\pi$$

to obtain

$$v_R = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

We can leave the argument of the sine in this form for convenience, or we can write it as (377 rad/s)t or as (377 s^{-1})t.

(b) What are the current $i_R(t)$ in the resistance and the amplitude I_R of $i_R(t)$?

KEY IDEA

In an ac circuit with a purely resistive load, the alternating current $i_R(t)$ in the resistance is *in phase* with the alternating potential difference $v_R(t)$ across the resistance; that is, the phase constant ϕ for the current is zero.

Calculations: Here we can write Eq. 31-29 as

$$i_R = I_R \sin(\omega_d t - \phi) = I_R \sin \omega_d t. \qquad (31\text{-}35)$$

From Eq. 31-33, the amplitude I_R is

$$I_R = \frac{V_R}{R} = \frac{36.0 \text{ V}}{200\ \Omega} = 0.180 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-35, we have

$$i_R = (0.180 \text{ A}) \sin(120\pi t). \qquad \text{(Answer)}$$

31.8: Three Simple Circuits:
ii. A Capacitive Load:

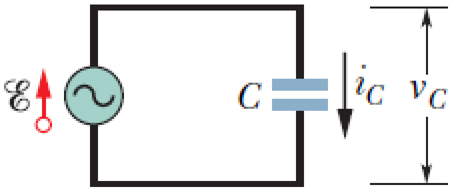


Fig. 31-10 A capacitor is connected across an alternating-current generator.

$$v_C = V_C \sin \omega_d t,$$

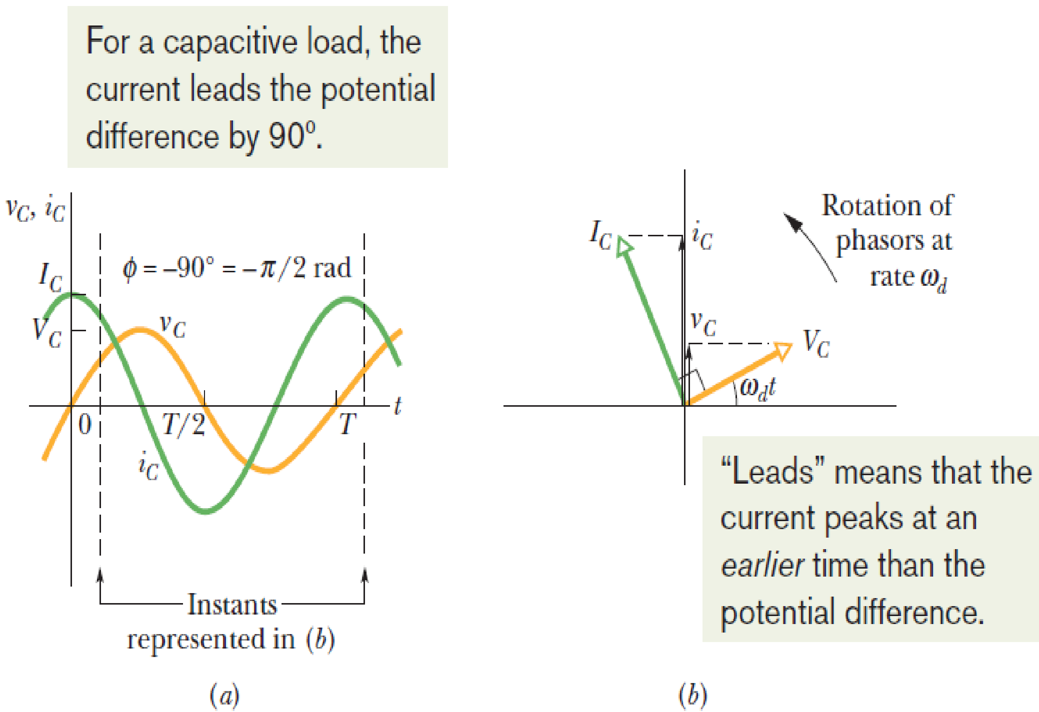
$$q_C = C v_C = C V_C \sin \omega_d t.$$

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t.$$

$$X_C = \frac{1}{\omega_d C} \qquad \text{(capacitive reactance).}$$

X_C is called the **capacitive reactance of a capacitor.** The SI unit of X_C is the *ohm*, just as for resistance *R*.

31.8: Three Simple Circuits:
ii. A Capacitive Load:



$$\cos \omega_d t = \sin(\omega_d t + 90°).$$

$$i_C = \left(\frac{V_C}{X_C}\right) \sin(\omega_d t + 90°).$$

$$i_C = I_C \sin(\omega_d t - \phi),$$

$$V_C = I_C X_C \qquad \text{(capacitor).}$$

Fig. 31-11 (*a*) The current in the capacitor leads the voltage by 90° (= $\pi/2$ rad). (*b*) A phasor diagram shows the same thing.

Example, Purely capacitive load: potential difference and current

In Fig. 31-10, capacitance C is 15.0 μF and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_C(t)$ across the capacitance and the amplitude V_C of $v_C(t)$?

KEY IDEA

In a circuit with a purely capacitive load, the potential difference $v_C(t)$ across the capacitance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here we have $v_C(t) = \mathscr{E}(t)$ and $V_C = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we have

$$V_C = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_C(t)$, we use Eq. 31-28 to write

$$v_C(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (31\text{-}43)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-43, we have

$$v_C = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_C(t)$ in the circuit as a function of time and the amplitude I_C of $i_C(t)$?

KEY IDEA

In an ac circuit with a purely capacitive load, the alternating current $i_C(t)$ in the capacitance leads the alternating potential difference $v_C(t)$ by 90°; that is, the phase constant ϕ for the current is $-90°$, or $-\pi/2$ rad.

Calculations: Thus, we can write Eq. 31-29 as

$$i_C = I_C \sin(\omega_d t - \phi) = I_C \sin(\omega_d t + \pi/2). \qquad (31\text{-}44)$$

We can find the amplitude I_C from Eq. 31-42 ($V_C = I_C X_C$) if we first find the capacitive reactance X_C. From Eq. 31-39 ($X_C = 1/\omega_d C$), with $\omega_d = 2\pi f_d$, we can write

$$X_C = \frac{1}{2\pi f_d C} = \frac{1}{(2\pi)(60.0 \text{ Hz})(15.0 \times 10^{-6} \text{ F})}$$
$$= 177\ \Omega.$$

Then Eq. 31-42 tells us that the current amplitude is

$$I_C = \frac{V_C}{X_C} = \frac{36.0 \text{ V}}{177\ \Omega} = 0.203 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-44, we have

$$i_C = (0.203 \text{ A}) \sin(120\pi t + \pi/2). \qquad \text{(Answer)}$$

31.8: Three Simple Circuits:
iii. An Inductive Load:

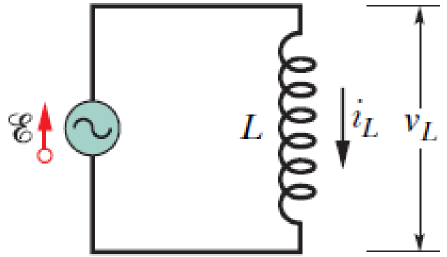


Fig. 31-12 An inductor is connected across an alternating-current generator.

$$v_L = V_L \sin \omega_d t, \qquad v_L = L\frac{di_L}{dt}.$$

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin \omega_d t.$$

$$i_L = \int di_L = \frac{V_L}{L}\int \sin \omega_d t\, dt = -\left(\frac{V_L}{\omega_d L}\right)\cos \omega_d t.$$

$$X_L = \omega_d L \qquad \text{(inductive reactance).}$$

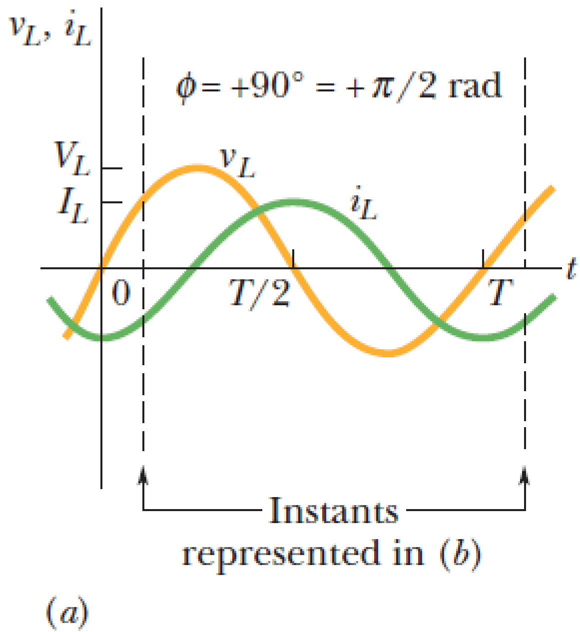
$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90^\circ). \qquad i_L = I_L \sin(\omega_d t - \phi),$$

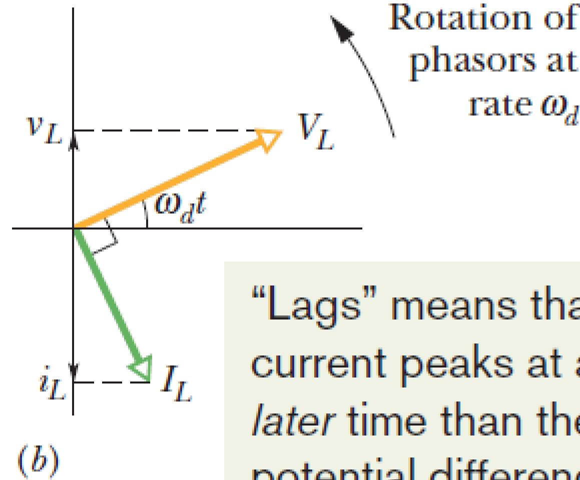
$$V_L = I_L X_L \qquad \text{(inductor).}$$

The value of X_L , **the inductive resistance**, depends on the driving angular frequency ω_d.
The unit of the inductive time constant τ_L indicates that the SI unit of X_L is the *ohm*.

31.8: Three Simple Circuits:
iii. An Inductive Load:

For an inductive load, the current lags the potential difference by 90º.





"Lags" means that the current peaks at a *later* time than the potential difference.

Fig. 31-13 (*a*) The current in the inductor lags the voltage by 90° (= $\pi/2$ rad). (*b*) A phasor diagram shows the same thing.

Example, Purely inductive load: potential difference and current

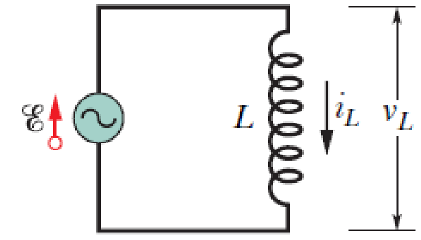


Fig. 31-12 An inductor is connected across an alternating-current generator.

In Fig. 31-12, inductance L is 230 mH and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_L(t)$ across the inductance and the amplitude V_L of $v_L(t)$?

KEY IDEA

In a circuit with a purely inductive load, the potential difference $v_L(t)$ across the inductance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here we have $v_L(t) = \mathscr{E}(t)$ and $V_L = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we know that

$$V_L = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_L(t)$, we use Eq. 31-28 to write

$$v_L(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (31\text{-}53)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-53, we have

$$v_L = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_L(t)$ in the circuit as a function of time and the amplitude I_L of $i_L(t)$?

KEY IDEA

In an ac circuit with a purely inductive load, the alternating current $i_L(t)$ in the inductance lags the alternating potential difference $v_L(t)$ by 90°. (In the mnemonic of the problem-solving tactic, this circuit is "positively an *ELI* circuit," which tells us that the emf E leads the current I and that ϕ is *positive*.)

Calculations: Because the phase constant ϕ for the current is $+90°$, or $+\pi/2$ rad, we can write Eq. 31-29 as

$$i_L = I_L \sin(\omega_d t - \phi) = I_L \sin(\omega_d t - \pi/2). \qquad (31\text{-}54)$$

We can find the amplitude I_L from Eq. 31-52 ($V_L = I_L X_L$) if we first find the inductive reactance X_L. From Eq. 31-49 ($X_L = \omega_d L$), with $\omega_d = 2\pi f_d$, we can write

$$X_L = 2\pi f_d L = (2\pi)(60.0 \text{ Hz})(230 \times 10^{-3} \text{ H}) = 86.7\ \Omega.$$

Then Eq. 31-52 tells us that the current amplitude is

$$I_L = \frac{V_L}{X_L} = \frac{36.0 \text{ V}}{86.7\ \Omega} = 0.415 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-54, we have

$$i_L = (0.415 \text{ A}) \sin(120\pi t - \pi/2). \qquad \text{(Answer)}$$

31.8: Three Simple Circuits:

Table 31-2

Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	$0°$ (= 0 rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by $90°$ (= $\pi/2$ rad)	$-90°$ (= $-\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $90°$ (= $\pi/2$ rad)	$+90°$ (= $+\pi/2$ rad)	$V_L = I_L X_L$

31.9: The Series RLC Circuit:

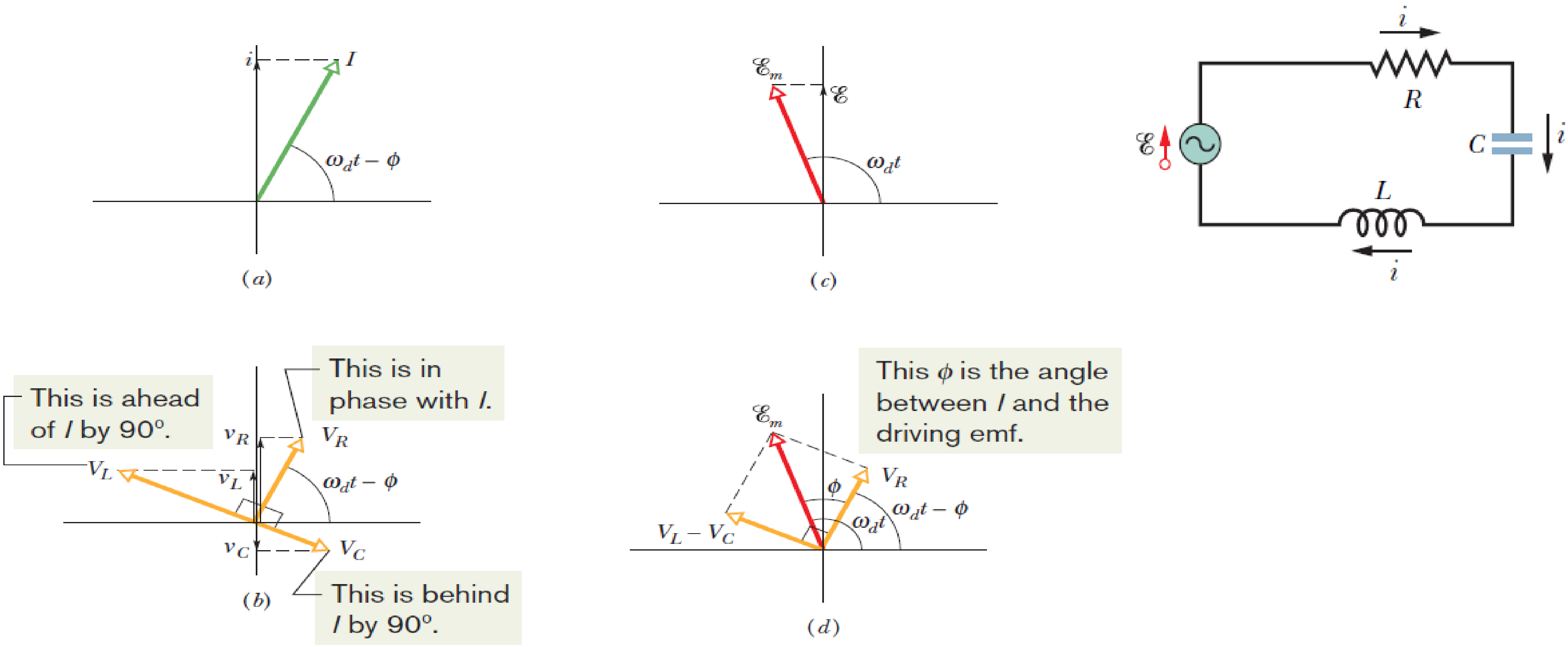
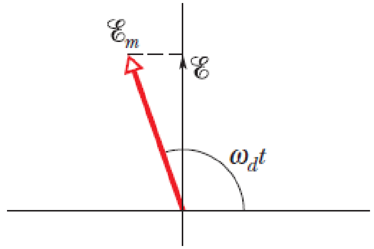


Fig. 31-14 (*a*) A phasor representing the alternating current in the driven *RLC* circuit at time *t*. The amplitude *I*, the instantaneous value *i*, and the phase($\omega_d t$-ϕ) are shown.
(*b*) Phasors representing the voltages across the inductor, resistor, and capacitor, oriented with respect to the current phasor in (*a*).
(*c*) A phasor representing the alternating emf that drives the current of (*a*).
(*d*) The emf phasor is equal to the vector sum of the three voltage phasors of (*b*).Here, voltage phasors V_L and V_C have been added vectorially to yield their net phasor (V_L-V_C).

31.9: The Series RLC Circuit:

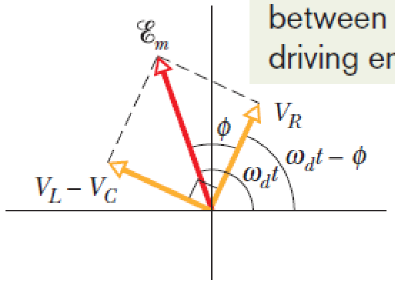


$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t$$

$$i = I \sin(\omega_d t - \phi)$$

$$\mathscr{E}_m^2 = V_R^2 + (V_L - V_C)^2. = (IR)^2 + (IX_L - IX_C)^2,$$

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}.$$



$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance defined).}$$

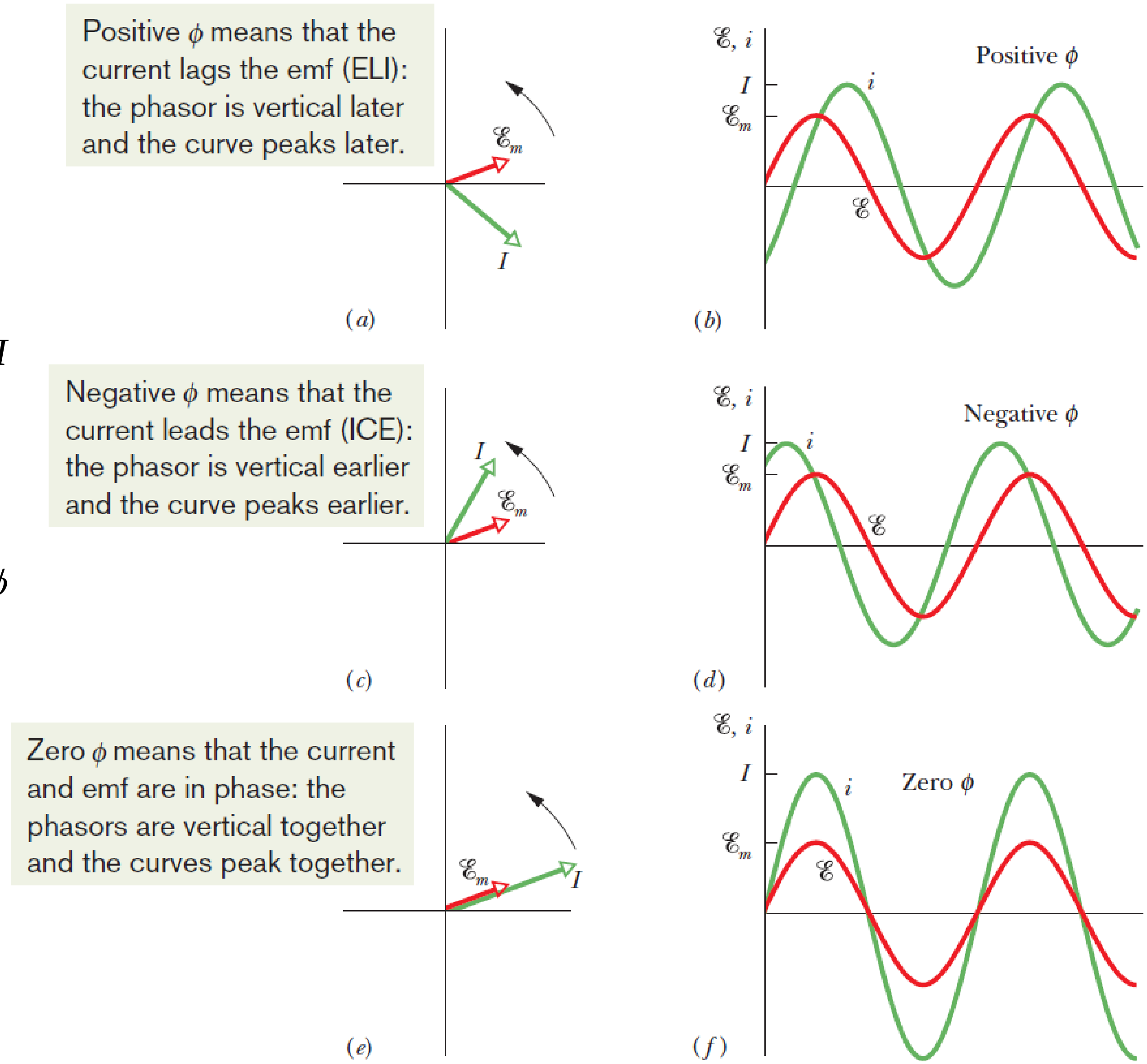
$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude).}$$

$$\tan \phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR},$$

$$\tan \phi = \frac{X_L - X_C}{R} \quad \text{(phase constant).}$$

31.9: The Series RLC Circuit:

Fig. 31-15 Phasor diagrams and graphs of the alternating emf and current *i* for the driven *RLC* circuit. In the phasor diagram of *(a)* and the graph of *(b)*, the current *I* lags the driving emf and the current's phase constant ϕ is positive. In *(c)* and *(d)*, the current *i* leads the driving emf and its phase constant ϕ is negative. In *(e)* and *(f)*, the current *i* is in phase with the driving emf and its phase constant ϕ is zero.



31.9: The Series RLC Circuit, Resonance:

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \qquad \text{(current amplitude).}$$

For a given resistance *R*, that amplitude is a maximum when the quantity ($\omega_d L$ -$1/\omega_d C$) in the denominator is zero.

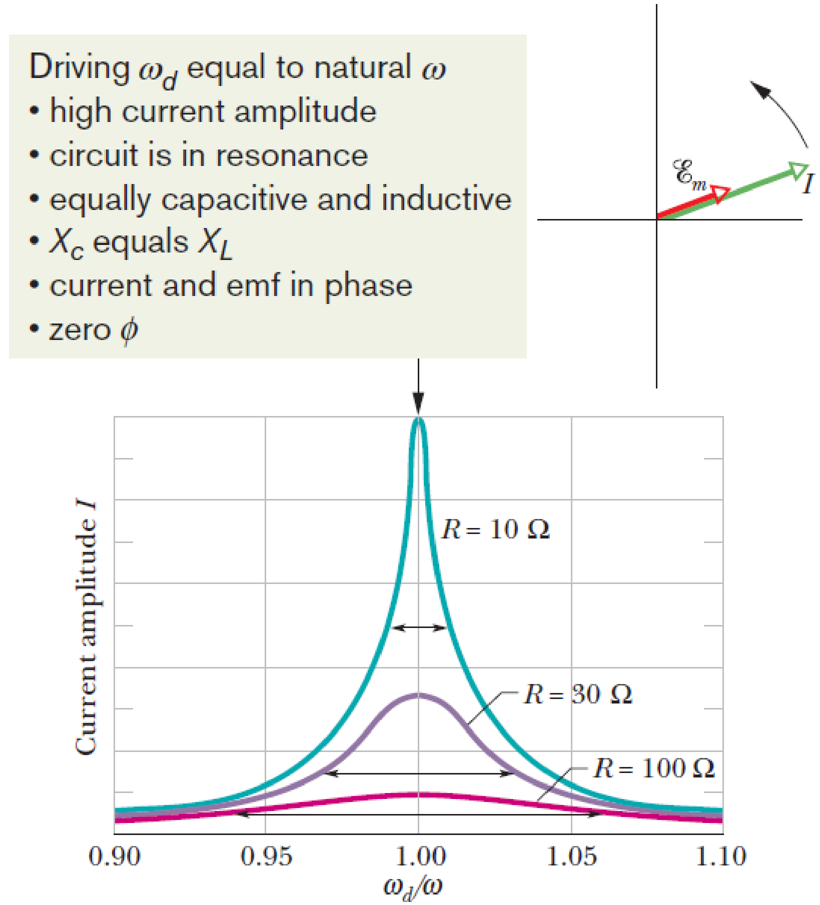
$$\Rightarrow \quad \omega_d L = \frac{1}{\omega_d C} \quad \Rightarrow \quad \omega_d = \frac{1}{\sqrt{LC}} \qquad \text{(maximum } I\text{).}$$

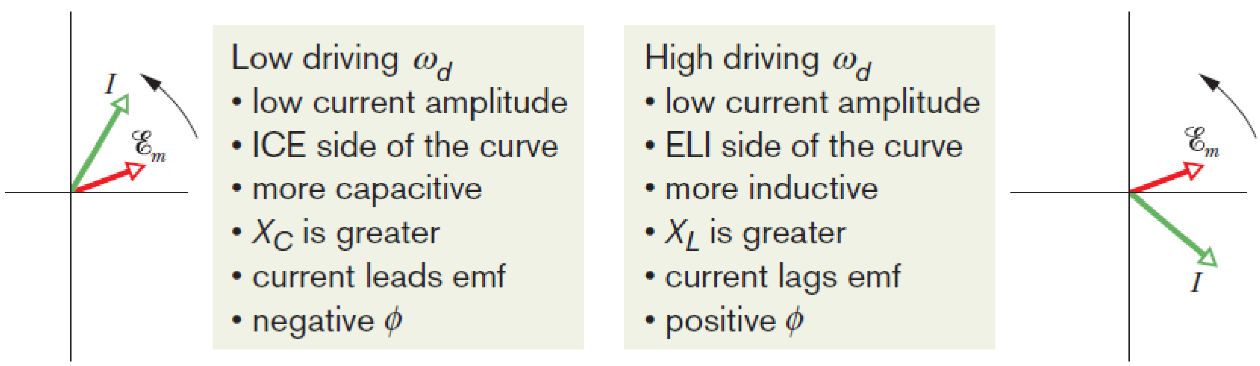
The maximum value of *I* occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance.

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \qquad \text{(resonance).}$$

31.9: The Series RLC Circuit, Resonance:

Fig. 31-16 *Resonance curves* for the driven *RLC* circuit of Fig. 31-7 with $L = 100\ \mu H$, $C = 100$ pF, and three values of R. The current amplitude I of the alternating current depends on how close the driving angular frequency ω_d is to the natural angular frequency ω. The horizontal arrow on each curve measures the curve's *half-width*, which is the width at the half-maximum level and is a measure of the sharpness of the resonance. To the left of $\omega_d/\omega = 1.00$, the circuit is mainly capacitive, with $X_C > X_L$; to the right, it is mainly inductive, with $X_L > X_C$.





31.10: Power in Alternating-Current Circuits:

The instantaneous rate at which energy is dissipated in the resistor:

$$P = i^2R = [I\sin(\omega_d t - \phi)]^2 R = I^2 R \sin^2(\omega_d t - \phi).$$

The average rate at which energy is dissipated in the resistor, is the average of this over time:

$$P_{avg} = \frac{I^2R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R.$$

Since the root mean square of the current is given by:

Similarly, $I_{rms} = \frac{I}{\sqrt{2}}$ ➡ $P_{avg} = I_{rms}^2 R$ (average power).

With $V_{rms} = \frac{V}{\sqrt{2}}$ and $\mathscr{E}_{rms} = \frac{\mathscr{E}_m}{\sqrt{2}}$ (rms voltage; rms emf).

Therefore, $I_{rms} = \frac{\mathscr{E}_{rms}}{Z} = \frac{\mathscr{E}_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}$,

➡ $$P_{avg} = \frac{\mathscr{E}_{rms}}{Z} I_{rms} R = \mathscr{E}_{rms} I_{rms} \frac{R}{Z}.$$

➡ $P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi$ (average power), where $\cos\phi = \frac{V_R}{\mathscr{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}.$

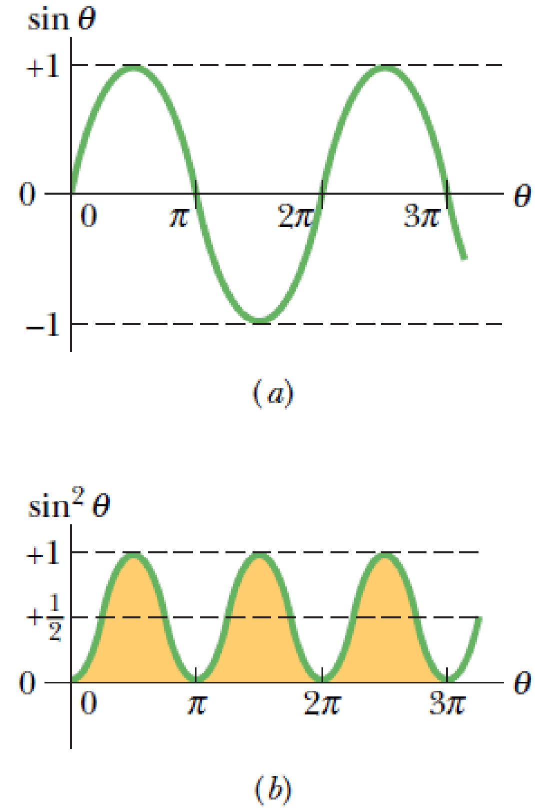


Fig. 31-17 (a) A plot of sin θ versus θ. The average value over one cycle is zero. (b) A plot of $\sin^2\theta$ versus θ. The average value over one cycle is $\frac{1}{2}$.

Example, Driven RLC circuit:

A series RLC circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(a) What are the power factor $\cos\phi$ and phase constant ϕ of the circuit?

KEY IDEA

The power factor $\cos\phi$ can be found from the resistance R and impedance Z via Eq. 31-75 ($\cos\phi = R/Z$).

Calculations: To calculate Z, we use Eq. 31-61:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{(200\ \Omega)^2 + (80.0\ \Omega - 150\ \Omega)^2} = 211.90\ \Omega.$$

Equation 31-75 then gives us

$$\cos\phi = \frac{R}{Z} = \frac{200\ \Omega}{211.90\ \Omega} = 0.9438 \approx 0.944. \qquad \text{(Answer)}$$

Taking the inverse cosine then yields

$$\phi = \cos^{-1} 0.944 = \pm 19.3°.$$

Both $+19.3°$ and $-19.3°$ have a cosine of 0.944. To determine which sign is correct, we must consider whether the current leads or lags the driving emf. Because $X_C > X_L$, this circuit is mainly capacitive, with the current leading the emf. Thus, ϕ must be negative:

$$\phi = -19.3°. \qquad \text{(Answer)}$$

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

KEY IDEAS

There are two ways and two ideas to use: (1) Because the circuit is assumed to be in steady-state operation, the rate at which energy is dissipated in the resistance is equal to the rate at which energy is supplied to the circuit, as given by Eq. 31-76 ($P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi$). (2) The rate at which energy is dissipated in a resistance R depends on the square of the rms current I_{rms} through it, according to Eq. 31-71 ($P_{avg} = I^2_{rms} R$).

$$P_{avg} = I^2_{rms} R = \frac{\mathscr{E}^2_{rms}}{Z^2} R$$
$$= \frac{(120\ \text{V})^2}{(211.90\ \Omega)^2}(200\ \Omega) = 64.1\ \text{W}. \qquad \text{(Answer)}$$

Example, Driven RLC circuit, cont.:

A series RLC circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(c) What new capacitance C_{new} is needed to maximize P_{avg} if the other parameters of the circuit are not changed?

KEY IDEAS

(1) The average rate P_{avg} at which energy is supplied and dissipated is maximized if the circuit is brought into resonance with the driving emf. (2) Resonance occurs when $X_C = X_L$.

Calculations: From the given data, we have $X_C > X_L$. Thus, we must decrease X_C to reach resonance. From Eq. 31-39 ($X_C = 1/\omega_d C$), we see that this means we must increase C to the new value C_{new}.

Using Eq. 31-39, we can write the resonance condition $X_C = X_L$ as

$$\frac{1}{\omega_d C_{new}} = X_L.$$

Substituting $2\pi f_d$ for ω_d (because we are given f_d and not ω_d) and then solving for C_{new}, we find

$$C_{new} = \frac{1}{2\pi f_d X_L} = \frac{1}{(2\pi)(60\text{ Hz})(80.0\ \Omega)}$$

$$= 3.32 \times 10^{-5}\text{ F} = 33.2\ \mu\text{F}. \qquad \text{(Answer)}$$

31.11: Transformers:

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory).

Nobody wants an electric toaster or a child's electric train to operate at, say, 10 kV.

On the other hand, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize I^2R losses (often called ohmic losses) in the transmission line.

31.11: Transformers:

A device with which we can raise and lower the ac voltage in a circuit, keeping the product current voltage essentially constant, is called the **transformer.**

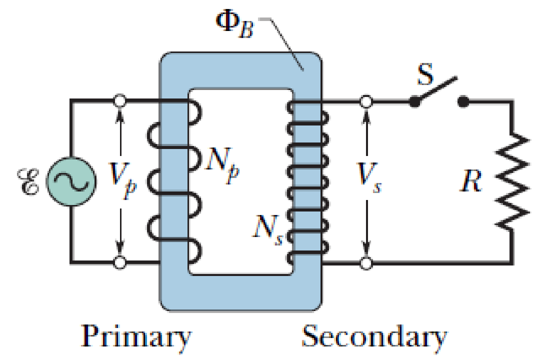


Fig. 31-18 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load *R* when switch S is closed.

The ideal transformer consists of two coils, with different numbers of turns, wound around an iron core.

In use, the primary winding, of N_p turns, is connected to an alternating-current generator whose emf at any time *t* is given by $\mathscr{E} = \mathscr{E}_m \sin \omega t.$

The secondary winding, of N_s turns, is connected to load resistance *R*, but its circuit is an open circuit as long as switch S is open.

The small sinusoidally changing primary current I_{mag} produces a sinusoidally changing magnetic flux *B* in the iron core.

Because *B* varies, it induces an emf (*dB/dt)* in each turn of the secondary. This emf per turn is the same in the primary and the secondary. Across the primary, the voltage $V_p = \mathcal{E}_{turn} N_p$. Similarly, across the secondary the voltage is $V_s = \mathcal{E}_{turn} N_s$.

$$V_s = V_p \frac{N_s}{N_p} \quad \text{(transformation of voltage).}$$

31.11: Transformers:

$$V_s = V_p \frac{N_s}{N_p} \quad \text{(transformation of voltage).}$$

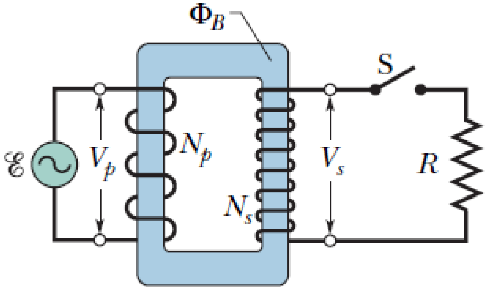


Fig. 31-18 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

If $N_s > N_p$, the device is a step-up transformer because it steps the primary's voltage V_p up to a higher voltage V_s. Similarly, if $N_s < N_p$, it is a step-down transformer.

If no energy is lost along the way, conservation of energy requires that

$$I_p V_p = I_s V_s. \quad \Rightarrow \quad I_s = I_p \frac{N_p}{N_s} \quad \text{(transformation of currents).}$$

$$\Rightarrow \quad I_p = \frac{1}{R}\left(\frac{N_s}{N_p}\right)^2 V_p. \quad \Rightarrow \quad R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R.$$

Here R_{eq} is the value of the load resistance as "seen" by the generator.

For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. For ac circuits, for the same to be true, the *impedance* (rather than just the resistance) of the generator must equal that of the load.

Example, Transformer:

A transformer on a utility pole operates at $V_p = 8.5$ kV on the primary side and supplies electrical energy to a number of nearby houses at $V_s = 120$ V, both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load, and a power factor of unity.

(a) What is the turns ratio N_p/N_s of the transformer?

KEY IDEA

The turns ratio N_p/N_s is related to the (given) rms primary and secondary voltages via Eq. 31-79 ($V_s = V_p N_s/N_p$).

Calculation: We can write Eq. 31-79 as

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \qquad (31\text{-}83)$$

(Note that the right side of this equation is the *inverse* of the turns ratio.) Inverting both sides of Eq. 31-83 gives us

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3 \text{ V}}{120 \text{ V}} = 70.83 \approx 71. \qquad \text{(Answer)}$$

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is 78 kW. What are the rms currents in the primary and secondary of the transformer?

Calculations: In the primary circuit, with $V_p = 8.5$ kV, Eq. 31-77 yields

$$I_p = \frac{P_{avg}}{V_p} = \frac{78 \times 10^3 \text{ W}}{8.5 \times 10^3 \text{ V}} = 9.176 \text{ A} \approx 9.2 \text{ A}. \qquad \text{(Answer)}$$

Similarly, in the secondary circuit,

$$I_s = \frac{P_{avg}}{V_s} = \frac{78 \times 10^3 \text{ W}}{120 \text{ V}} = 650 \text{ A}. \qquad \text{(Answer)}$$

You can check that $I_s = I_p(N_p/N_s)$ as required by Eq. 31-80.

(c) What is the resistive load R_s in the secondary circuit? What is the corresponding resistive load R_p in the primary circuit?

$$R_s = \frac{V_s}{I_s} = \frac{120 \text{ V}}{650 \text{ A}} = 0.1846\ \Omega \approx 0.18\ \Omega. \qquad \text{(Answer)}$$

Similarly, for the primary circuit we find

$$R_p = \frac{V_p}{I_p} = \frac{8.5 \times 10^3 \text{ V}}{9.176 \text{ A}} = 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$